

University of Groningen

No Longer Discrete

Niezink, Nynke M. D.; Snijders, Tom A. B.; van Duijn, Marijtje A. J.

Published in:
 Sociological Methodology

DOI:
[10.1177/0081175019842263](https://doi.org/10.1177/0081175019842263)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
 Publisher's PDF, also known as Version of record

Publication date:
 2019

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Niezink, N. M. D., Snijders, T. A. B., & van Duijn, M. A. J. (2019). No Longer Discrete: Modeling the Dynamics of Social Networks and Continuous Behavior. *Sociological Methodology*, 49(1), 295-340. <https://doi.org/10.1177/0081175019842263>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.



NO LONGER DISCRETE: MODELING THE DYNAMICS OF SOCIAL NETWORKS AND CONTINUOUS BEHAVIOR

*Nynke M. D. Niezink**
Tom A. B. Snijders,†*
*Marijtje A. J. van Duijn**

Abstract

The dynamics of individual behavior are related to the dynamics of the social structures in which individuals are embedded. This implies that in order to study social mechanisms such as social selection or peer influence, we need to model the evolution of social networks and the attributes of network actors as interdependent processes. The stochastic actor-oriented model is a statistical approach to study network-attribute coevolution based on longitudinal data. In its standard specification, the coevolving actor attributes are assumed to be measured on an ordinal categorical scale. Continuous variables first need to be discretized to fit into such a modeling framework. This article presents an extension of the stochastic actor-oriented model that does away with this restriction by using a stochastic differential equation to model the evolution of a continuous attribute. We propose a measure for explained variance and give an interpretation of parameter sizes. The proposed method

*University of Groningen, Groningen, The Netherlands

†University of Oxford, Oxford, UK

Corresponding Author:

Nynke M. D. Niezink, Department of Statistics and Data Science, Carnegie Mellon University, Baker Hall, Pittsburgh, PA 15213, USA.

Email: nniezink@andrew.cmu.edu

is illustrated by a study of the relationship between friendship, alcohol consumption, and self-esteem among adolescents.

Keywords

social networks, longitudinal data, continuous-time modeling, stochastic differential equations

1. INTRODUCTION

Social actors, such as people, organizations, and countries, simultaneously shape and are shaped by their social context. For example, individuals may select their friends based on their behavior, but they may also change their own behavior based on that of their friends. Social relationships among actors, such as friendship or collaboration, can be represented in social networks. These networks can change over time, often in interdependent relation with changing actor characteristics. While earlier studies of network dynamics were mainly descriptive in nature, many statistical models for network dynamics have been developed since (e.g., Almquist and Butts 2014; Doreian and Stokman 1997; Krivitsky and Handcock 2014; Robins and Pattison 2001; Snijders 2001, 2005). For the study of the interdependent dynamics of networks and individual behavior, the stochastic actor-oriented model is widely used (Snijders, Steglich, and Schweinberger 2007; Steglich, Snijders, and Pearson 2010).

The stochastic actor-oriented model can be used to test hypotheses about the social mechanisms driving network and actor attribute dynamics and study the interdependent processes of partner selection and social influence. One basic assumption of the model proposed by Snijders et al. (2007) is that dynamic actor attributes are measured on an ordinal scale with a limited number of categories. Under this assumption, the network and attribute evolution can be represented in a common statistical framework (i.e., by a continuous-time Markov chain with a discrete outcome space). However, restricting the coevolving attribute to a limited number of categories has proven to be a practical limitation in several studies because of the necessity to discretize attributes measured on a very fine-grained or continuous scale.

In a study of the development of body weight of adolescents and their friendships, for example, Haye et al. (2011) split the dependent attribute, body mass index, into ordered categories to make their analysis feasible.

Flashman (2012) measured scholastic achievement on a continuous scale and later transformed it into a five-point scale for the same purpose. Some other continuous variables that had to be treated similarly are job satisfaction (Agneessens and Wittek 2008), self-reported and peer-reported aggression and victimization (Dijkstra et al. 2012), and physical activity (Gesell, Tesdahl, and Ruchman 2012).

For corporate actors, performance indicators composed of multiple variables and monetary outcomes are often measured on a continuous scale. Many individual physical attributes are continuous variables. Psychological scales often assume the existence of one or more latent continuous dimensions, measured on a fine-grained categorical scale. For all such measures, discretization into a few categories would involve arbitrary choices (number and width of categories) and could lead to loss of information. Moreover, different discretizations could lead to different results.

The role of peers for organizational performance or individual health outcomes has been the subject of several studies (e.g., Checkley et al. 2014; Haye et al. 2011). Psychological characteristics may be susceptible to social influence (e.g., depression through corumination), but they may also play a buffering role. People with certain psychological characteristics might be more susceptible to influence by their peers than others. If we want to study such a moderating effect, unless the characteristic is a stable personality trait, we need to model the network through which influence occurs, the variable that is influenced, and the psychological moderator as three coevolving elements.

Motivated by the practical limitations discussed previously, Niezink and Snijders (2017) have extended the stochastic actor-oriented model for the coevolution of networks and continuous actor behavior. In this article, we first give an applied introduction to this model. Newly treated in the current study are a proposal for defining explained variance and an extensive discussion about interpretation of results, two topics that are not addressed in Niezink and Snijders (2017) but that are essential for the model to be useful in practice. We use the method to study whether self-esteem moderates adolescents' susceptibility to peer influence on alcohol use by combining a stochastic actor-oriented model for the coevolution of a network and a discrete individual behavior variable (Steglich et al. 2010) with the model for continuous actor behavior (Niezink and Snijders 2017). The study illustrates how the new model may be of help in moving forward and transcending the "selection

versus influence” narrative that has gained popularity after Steglich et al. (2010).

The model for the coevolution of networks and continuous actor behavior applies a stochastic differential equation to model behavior dynamics. Stochastic differential equations model the dynamics of continuous variables and can be considered the continuous-time version of autoregressive models for time series data. The use of ordinary differential equations, their deterministic counterparts, in sociological applications was first advocated by Coleman (1964, 1968). Such models quickly became a standard part of the toolbox of mathematical sociologists (Beltrami 1993; Blalock 1969). Applications include the study of inequality in socioeconomic careers (Rosenfeld and Nielsen 1984) and the study of change in academic achievement and the role of school effects in this process (Sørensen 1996).

Stochastic differential equations have been applied extensively in econometrics and financial mathematics (e.g., Fouque, Papanicolaou, and Sinclair 2000), but many of the contributions to the social science literature have been primarily technical (e.g., Bergstrom 1984; Hamerle, Singer, and Nagl 1993; Oud and Jansen 2000; Singer 1998, 2012). However, recent work attests to an increase in interest in the application of stochastic differential equations in the social sciences (e.g., Deboeck 2012; Oravecz, Tuerlinckx, and Vandekerckhove 2011; Reinecke, Schmidt, and Weick 2005; Voelkle et al. 2012). Moreover, their use is stimulated through the introduction of open source software (e.g., Driver, Oud, and Voelkle 2017).

Relational phenomena too have been studied by differential equations but mainly on the dyad level—that is, pertaining to pairs of individuals. Nicholson et al. (2011), for example, assessed the reciprocal relationships between maternal depressive symptoms and children’s behaviors. Felmlee and Greenberg (1999) defined a theoretical model of romantic partner interaction using differential equations in which the behaviors of partners mutually affect each other. This model has been applied to study affective dynamics in couples (e.g., Steele, Ferrer, and Nesselroade 2014).

So far, stochastic differential equations in the social sciences have mainly been applied to study developmental processes within individuals or in dyads from a psychological perspective. In this article, stochastic differential equation models are combined with models for the

evolution of social structures, opening them up to a new world of sociological questions.

One sociological puzzle is that of network autocorrelation—the phenomenon that in a social network, related social actors often show similarities. Social influence and network partner selection based on shared characteristics are examples of processes that may lead to network autocorrelation. Many methods have been proposed to study social influence on continuous behavior variables. Reviews of methods to identify peer effects include Mouw (2006), An (2014), and Sacerdote (2014). In several models, social influence is studied while keeping the network fixed. For example, the linear model in which actors' attribute values are regressed on the average value of their network neighbors constitutes a basic statistical model for peer influence (Manski 1993). A related model for social influence is the network autocorrelation model, which originates in spatial analysis as a linear model for spatially distributed data (e.g., Doreian 1980, 1981; Dow, Burton, and White 1982; Friedkin 1998; Leenders 2002; Ord 1975). While originally this model was defined for cross-sectional data, extensions for longitudinal data have been proposed as well (e.g., Cressie 1991; Elhorst 2001; Zhu et al. 2017).

Steglich et al. (2010) argue that to come to grips with the distinction between selection and influence, a case of the “reflection problem” as discussed by Manski (1993), it is necessary to study the mutual dependence between network and behavior, where both are studied longitudinally as endogenously changing structures. To study the simultaneous dynamics of spatial weights and continuous behavior variables, Hays, Kachi, and Franzese (2010) proposed an extension of the spatial autocorrelation model in which the spatial weights are estimated based on covariates explaining connectivity and on the dynamic individual behavior variables. O'Malley (2013) touched on the idea of combining a temporal network autocorrelation model with a temporal extension of the p_2 model, which is a dyad-independent random effects model for network data (Van Duijn, Snijders, and Zijlstra 2004). Leenders (1997) combined a longitudinal autocorrelation model with a dyad-independent Markov model for network dynamics.

While the longitudinal autocorrelation models mentioned previously are all discrete-time models, the model presented in this article is a continuous-time model extending the approach of stochastic actor-oriented models (Snijders 2001). The idea to use continuous-time

models for network evolution was already advocated by Holland and Leinhardt (1977) and Wasserman (1977). The advantages of continuous-time modeling are discussed, for example, by Voelkle et al. (2012) and Block et al. (2018). To summarize, continuous-time modeling provides a good framework for representing the feedback that is essential for interdependent dynamics (or coevolution) and offers a direct approach to overcome the problem of nonequidistant panel waves. Moreover, the model presented here does not assume dyad independence, and it can be used to study a wide range of social mechanisms, such as transitivity and popularity, driving network change. Such structural mechanisms cannot be modeled in the autocorrelation models mentioned previously.

The overall structure of this article is as follows. Section 2 briefly introduces stochastic differential equation models. Section 3 defines the model for the coevolution of a social network and the continuous attribute of network actors. Section 4 discusses effect sizes—parameter interpretation and explained variance—for the continuous attribute model. Section 5 presents the study of the coevolution of friendship, alcohol use, and self-esteem among adolescents.

2. STOCHASTIC DIFFERENTIAL EQUATIONS

This section briefly introduces stochastic differential equation models with a simple example. Øksendal (2000) and, in a more applied way, Iacus (2008) give general treatments of the topic.

A differential equation model is a continuous-time model describing the evolution of a continuous variable. In a continuous-time model, time is not an explanatory variable. Instead, the model as a whole, with time as an index variable, explains the dynamics underlying an evolutionary process (e.g., people do not change weight because of time but over time). The general form of an ordinary (i.e., nonstochastic) differential equation modeling the evolution of a variable z is

$$\frac{dz(t)}{dt} = f(z(t), u(t)). \quad (1)$$

We focus here only on first-order differential equations; that is, higher order derivatives are not included in the equation. The equation models the change in z , expressed by its derivative, as some function f of a set of explanatory variables u , which can be constant or time-dependent,

and the value of z itself. A simple example of an ordinary differential equation is

$$\frac{dz(t)}{dt} = az(t) + b. \quad (2)$$

The only function $z(t)$ that satisfies this differential equation is

$$z(t) = z_0 e^{at} + \frac{b}{a}(e^{at} - 1), \quad (3)$$

where z_0 denotes the value of z at time $t=0$. This function is the solution of equation 2. Parameter a is a feedback parameter; it represents the influence of $z(t)$ on its own rate of change. The stability of equation 3 is determined by feedback parameter a . Empirical growth processes are usually stable. However, the situation of explosive growth of social processes also has considerable theoretical interest (Sørensen 1978). Figure 1 illustrates the behavior of solutions to differential equation 2. If a is positive, $z(t)$ will increase (or decrease) at an ever-increasing rate, as shown in Figure 1a. If a is negative, $z(t)$ will converge to the equilibrium value $-b/a$ of the solution, as shown in Figure 1b. In the latter, stable case, $z(t)$ is the weighted mean of its initial value z_0 (weight e^a) and the equilibrium value $-b/a$ (weight $1 - e^a$) for any time t . The rate of change of $z(t)$ is proportional to the distance of $z(t)$ to the equilibrium. This means that values further away from the equilibrium move toward it at a faster pace than those close to the equilibrium (see again Figure 1b). Moreover, the larger $|a|$ is, the faster $z(t)$ will converge to the equilibrium value.

Differential equation 2 describes a deterministic process; given an initial value z_0 , it spells out the complete evolution of z . It also describes a very smooth process (see again Figure 1). In many applications, however, the evolution processes of the variables of interest behave erratically. In those cases, models that allow for random disturbances are more appropriate. Stochastic differential equation models do exactly this by including an error term in the differential equation (Øksendal 2000).

Let $Z(t)$ be a continuous random variable. A stochastic differential equation model, similar to the deterministic model 3, is

$$dZ(t) = [aZ(t) + b]dt + g dW(t), \quad Z(0) = z_0, \quad t \geq 0. \quad (4)$$

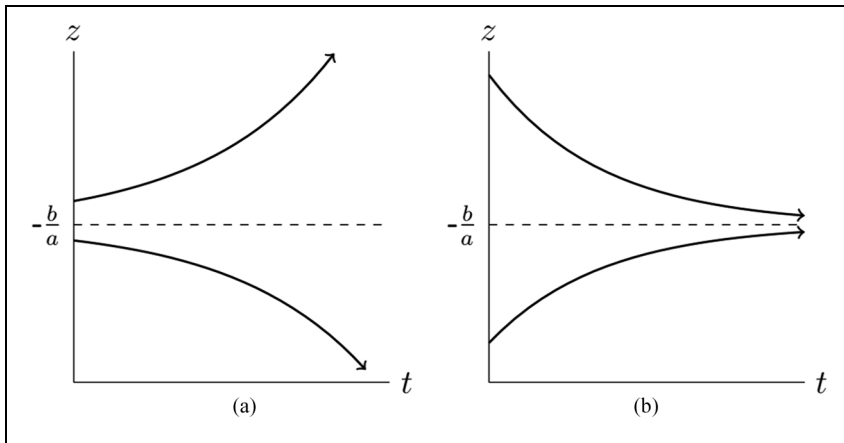


Figure 1. The behavior of solutions to differential equation 2.

Note. (a) The unstable situation: $a > 0$. (b) The stable situation: $a < 0$.

where $W(t)$ denotes the Wiener process (also known as standard Brownian motion), a continuous-time error process. The strength of the disturbance of $W(t)$ is given by diffusion coefficient g . A draw from a Wiener process is called a sample path. The Wiener process is characterized by the following three properties (e.g., Mikosch 1998):

1. Its initial value is zero: $W(0)=0$.
2. Its sample paths have no “jumps” (or more formally, $W(t)$ is continuous with probability one).
3. It has independent increments $W(t) - W(s)$ that are $\mathcal{N}(0, t - s)$ distributed, where $0 \leq s < t$.

Here, $\mathcal{N}(\mu, \nu)$ denotes a normal distribution with mean μ and variance ν . The independent increments property means that for all nonoverlapping time intervals $[s_1, t_1]$ and $[s_2, t_2]$, with $0 \leq s_1 < t_1 \leq s_2 < t_2$, the increments $W(t_1) - W(s_1)$ and $W(t_2) - W(s_2)$ are independent random variables, and the similar condition holds for any number of increments.

Combining the first and third property shows that $W(t)$ has a $\mathcal{N}(0, t)$ distribution for every $t \geq 0$. Both $W(t - s)$ and $W(t) - W(s)$ are normally distributed with mean zero and variance $t - s$. This variance is equal to the length $t - s$ of the interval under study. The larger the interval, the larger the fluctuations of the Wiener process on the interval. Note that even though the distributions of $W(t - s)$ and $W(t) - W(s)$

are the same, in a particular sample path, the subpath between 0 and $t - s$ and that between s and t do not need to look similar.

The Wiener process $W(t)$ is not differentiable with respect to time. Therefore, equation 4 is expressed in terms of infinitesimal increments $dZ(t)$ instead of the usual derivatives $dZ(t)/dt$. In fact, equation 4 is a shorthand notation for the stochastic integral equation

$$Z(t) = z_0 + \int_0^t [aZ(s) + b] ds + \int_0^t g dW(s), \quad (5)$$

where the second integral is an Itô stochastic integral (Øksendal 2000). An intuitive interpretation of equations 4 and 5 is that in a small time interval of length Δt , the stochastic process $Z(t)$ changes its value by an amount that is normally distributed with mean $[aZ(t) + b] \Delta t$ and variance $g^2 \Delta t$ and that is independent of the past behavior of the process. The solution to equations 4 and 5 is

$$Z(t) = z_0 e^{at} + \frac{b}{a} (e^{at} - 1) + g \int_0^t e^{a(t-s)} dW(s). \quad (6)$$

Unlike $z(t)$ in equation 3, $Z(t)$ is a random variable, normally distributed with mean

$$E(Z(t)) = z_0 e^{at} + \frac{b}{a} (e^{at} - 1) \quad (7)$$

and variance

$$\text{var}(Z(t)) = \frac{g^2}{2a} (e^{2at} - 1). \quad (8)$$

A derivation of equations 6 through 8 and further background on linear stochastic differential equations can be found in Mikosch (1998). In the case $a < 0$, for increasing values of t , the distribution of $Z(t)$ approaches a normal distribution with mean $-b/a$, as in the deterministic case, and variance $-g^2/(2a)$. This variance represents a balance between the diffusion coefficient g and the damping feedback a . Note that the additional stochastic term in differential equation 4 does not affect the expectation value, that is, $E(Z(t)) = z(t)$.

Figure 2 explores stochastic differential equation 4 with $a = -2$, $b = 6$, $g = 1$, and $z_0 = 0$. Figure 2a shows 50 sample paths (realizations) of the solution to the stochastic differential equation. The figure also

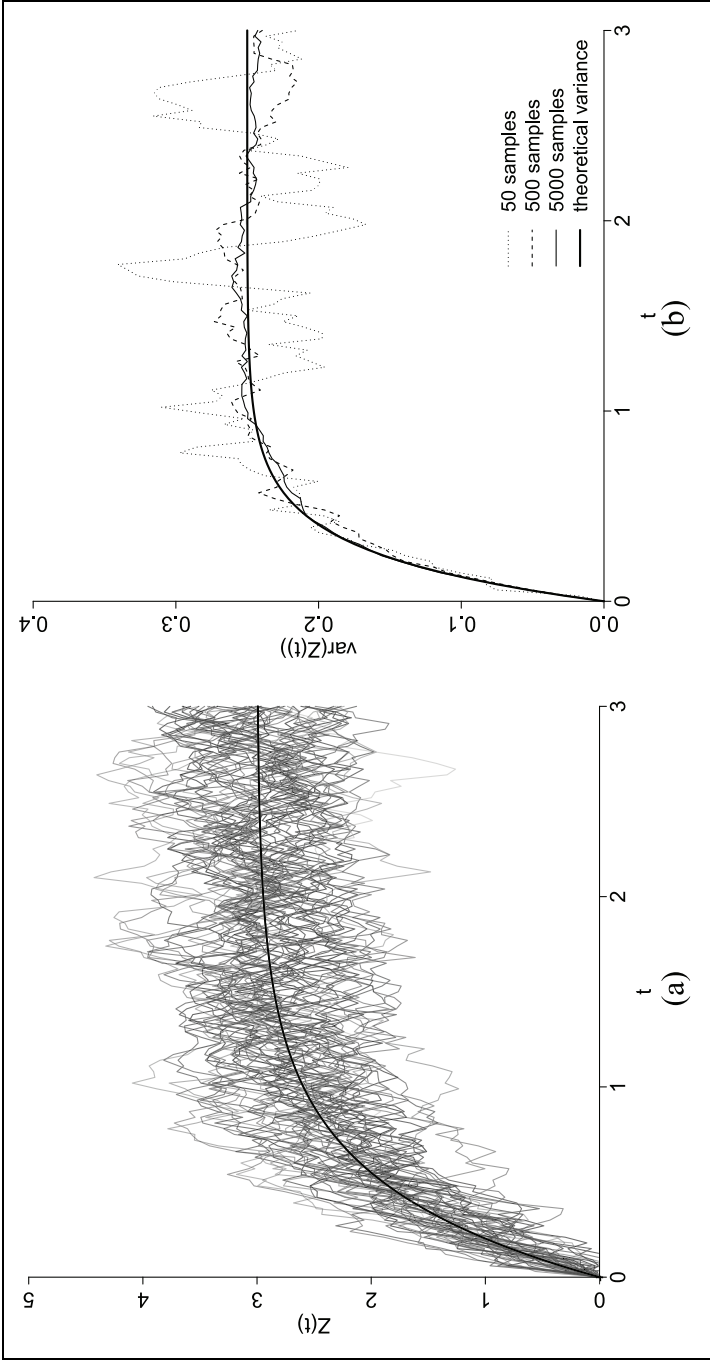


Figure 2. Exploring stochastic differential equation 4 with $a = -2$, $b = 6$, $g = 1$, and $z_0 = 0$.
Note. (a) The solution to ordinary differential equation 3 and 50 sample paths for stochastic differential equation 4. (b) The theoretical variance of $Z(t)$ as in equation 8 and the same variance based on 50, 500 and 5000 sample paths.

shows the solution of ordinary differential equation 2 for $a = -2$, $b = 6$, and $z_0 = 0$. The sample paths fluctuate around the solution of equation 2, which is in line with the fact that $E(Z(t)) = z(t)$. Figure 2b shows the variance of the 50 sample paths in Figure 2a. It also shows that when the number of sample paths is increased, their variance converges to the theoretical variance in equation 8. In the example, the equilibrium mean is 3, and the equilibrium variance is 0.25.

3. STOCHASTIC ACTOR-ORIENTED MODEL

The stochastic actor-oriented model represents network and attribute coevolution as an emergent group-level result of interdependent attribute changes and network changes (Snijders 2001; Snijders et al. 2007). One important characteristic of the model is its assumption that changes occur continuously in time. This means that in a real-valued time interval, changes can occur at any time point. The models discussed by Snijders (2001) and Snijders et al. (2007) are defined for discrete outcome spaces so that a change is always a discrete jump (one tie change or one category change in an attribute value) and on a finite interval only finitely many jumps will occur.

In the stochastic actor-oriented model, we assume that the observations of the network and actor attributes at discrete time points are the outcomes of an underlying continuous-time Markov process. In the current extension, we model the network evolution by a continuous-time Markov chain (Norris 1997) and the evolution of a continuous actor attribute by a stochastic differential equation. These model components are discussed in Sections 3.2 and 3.3. Both processes satisfy the Markov property: Given their current state, the probability distribution of future states of the processes is independent of their past states. The Markov chain and the stochastic differential equation together constitute the network-attribute coevolution model (Section 3.4). In Section 3.1, we introduce the necessary notation.

3.1. Notation and Data Structure

The outcome variables for which the coevolution model is defined are the dynamic network and the dynamic actor attributes. The network is defined by its node set $\{1, \dots, n\}$, representing the network actors, and the binary tie variables X_{ij} , representing a directed relation between

actors; $X_{ij} = 1$ and $X_{ij} = 0$, respectively, indicate the presence and absence of a tie from actor i to actor j . The relation is assumed to be nonreflexive, that is, $X_{ii} = 0$ for $i = 1, \dots, n$. The network as a whole is represented by the $n \times n$ adjacency matrix $X = (X_{ij})$.

The actor attributes are continuous variables, and they are measured on an interval scale. We will specify the stochastic actor-oriented model for a single coevolving continuous attribute, but extension to multivariate attributes is straightforward (Niezink and Snijders 2017). The vector Z contains the attribute variables for the n actors; Z_i denotes the attribute of actor i . Time dependence in the model is indicated by denoting $X = X(t)$ and $Z = Z(t)$.

The data we consider are network-attribute panel data; the network and the attribute data are collected at two or more points t_m in time. The period between t_m and t_{m+1} will be referred to as period m . The data are indicated by lowercase letters. We thus observe networks $x(t_m)$ and attributes $z(t_m)$. The stochastic model components are indicated by uppercase letters, where $X(t)$ denotes the network model and $Z(t)$ the attribute model. The state of the model is given by $Y(t) = (X(t), Z(t))$. Although they are often not mentioned explicitly, exogenous actor covariates and dyadic covariates (characteristics of pairs of actors) may also be part of the state $Y(t)$.

3.2. Network Evolution Model

We include here a short definition of the stochastic actor-oriented model. For a detailed discussion, see Snijders (2001, 2005, 2017b). A characteristic property of the model is its actor-oriented architecture. Changes in the network are modeled as choices made by actors about their outgoing ties. In other words, actors control the ties they send. We assume that at any given moment, all actors act conditionally independently of each other given the current state of the network and attributes of all actors. Moreover, actors are assumed to make only one tie change at a time. Similar to many other agent-based models, the model is based on local rules for actor behavior. It combines the strengths of agent-based simulation and statistical modeling (Snijders and Steglich 2015).

The stochastic actor-oriented model decomposes the network evolution process into two stochastic subprocesses. The first subprocess models the speed by which the network changes or, more precisely, the rate at which each actor in the network gets an opportunity to change one of

their outgoing ties. The second subprocess models the mechanisms that determine which particular tie is changed when the opportunity arises. In the following, we specify both subprocesses.

For each actor i , the waiting time until the next opportunity to make a tie change is exponentially distributed with a parameter given by a rate function λ_i . The waiting time until any of the actors makes a change is exponentially distributed with rate $\lambda_+ = \sum_{i=1}^n \lambda_i$. The rate function λ_i may depend on the state $Y(t)$. Here we assume period-dependent constant and equal rate functions for all actors: In period m , $\lambda_i = \lambda_m$ for all actors i . We thus assume homogeneous change activity across actors. The extension to nonconstant rate functions is straightforward and has been implemented (Ripley et al. 2018; Snijders 2005).

If actor i has the opportunity to make a network change, he or she can either choose to maintain the status quo or change a tie to one of the other actors. In terms of adjacency matrices, the set of potential new states comprises the current state x itself and the $n - 1$ matrices that deviate from x in exactly one nondiagonal element in row i . Let $x^{(\pm ij)}$ denote the adjacency matrix equal to x , in which entry x_{ij} is changed into $1 - x_{ij}$. By definition, let $x^{(\pm ii)} = x$. The adjacency matrix corresponding to the new network will thus be of the form $x^{(\pm ij)}$ with $j \in \{1, \dots, n\}$.

The choice of actor i depends on the so-called objective function $f_i(x, z)$ that takes into account the potential new network state, the current state of the attributes, and actor and dyadic covariates. The probability that actor i selects $x^{(\pm ij)}$ as the next network state is of the form

$$p(x^{(\pm ij)}) = \frac{\exp(f_i(x^{(\pm ij)}, z))}{\sum_{h=1}^n \exp(f_i(x^{(\pm ih)}, z))}, \quad (9)$$

also known as the multinomial logit model (McFadden 1974). This model can be obtained as the result of myopic stochastic optimization. The objective function is defined as a weighted sum of network effects $s_{ik}(x, z)$,

$$f_i(x, z) = \sum_k \beta_k s_{ik}(x, z). \quad (10)$$

Parameter β_k indicates the strength of the k th effect, controlling for all other effects in the objective function. The effects represent the actor-level mechanisms governing network change, as the effects in $u_i(t)$ in equation 11 do for attribute change. Steglich et al. (2010) and Ripley

et al. (2018) provide an overview of the many effects that are currently implemented for stochastic actor-oriented models. Basic examples are the outdegree effect (number of outgoing ties), the reciprocity effect (number of reciprocated ties), and the transitivity effect (number of transitive triplets). These model the general tendency of actors to form ties, their tendency to reciprocate ties, and their tendency toward transitive closure (e.g., “befriending the friends of my friends”). Effects may also depend on actor attributes or covariates. In this way, the differential tendency of actors with high attribute or covariate values to send (ego effect) or receive (alter effect) network ties can be assessed.

The model for the dynamics of discrete actor behavior is defined analogously to the network evolution model (Snijders et al. 2007), with a behavior rate function and objective function. In the discrete behavior model, once actors get the opportunity to change their behavior, they can either increase or decrease their attribute value by 1 or keep the value constant. Choice probabilities are defined by a multinomial logit model, as in equation 9. Like the network, the actors’ attribute values are modeled as evolving in the smallest steps possible.

3.3. Continuous Attribute Evolution Model

Stochastic differential equation 4 describes the change in an attribute, but it does not include any information on what may have brought this change about. In our model, the dynamics of the attribute of an actor i may depend on characteristics of i (e.g., individual covariates, network position) and characteristics of others in the network. We model these dependencies through the elements of input vector $u_i(t) = (u_{i0}(t), \dots, u_{ir}(t))$ in the stochastic differential equation

$$dZ_i(t) = [aZ_i(t) + b^\top u_i(t)]dt + g dW_i(t), \quad Z_i(t_1) = z_i(t_1). \quad (11)$$

If $u_i(t)$ itself does not depend on $Z_i(t)$, the solution to this equation—similar to equation 6—is given by

$$Z_i(t) = e^{a(t-t_1)} z_i(t_1) + \int_{t_1}^t e^{a(t-s)} b^\top u_i(s) ds + \int_{t_1}^t e^{a(t-s)} g dW_i(s). \quad (12)$$

The parameters in vector $b = (b_0, \dots, b_r)$ represent the strength of the effects in input $u_i(t)$. By default, the model includes the unit variable $u_{i0}(t) = 1$, which has a role equivalent to that of the intercept in a linear

regression model. Other effects may include constant actor attributes, such as gender or height. These variables, being, respectively, binary and continuous, may be included directly in the stochastic differential equation. Categorical actor attributes can be included through ways known from linear regression, for example using dummy coding. In subsection 3.3.1, we discuss the effect of network-dependent actor characteristics on attribute dynamics.

Equation 11 can be used to study the evolution of actor attributes between two measurements. However, when we are interested in modeling data from more than two measurement moments, we use the alternative specification,

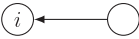
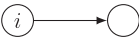


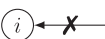
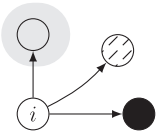
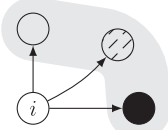
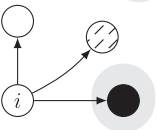
$$dZ_i(t) = \tau_m [a Z_i(t) + b^\top u_i(t)] dt + \sqrt{\tau_m} dW_i(t), \quad Z_i(t_m) = z_i(t_m), \quad (13)$$

where τ_m is a period-specific scale parameter, comparable to the rate parameter in the network evolution model. The scale parameters account for the fact that we model each period between consecutive measurements as having unit duration (Nieziink and Snijders 2017). The differences in period lengths are absorbed in the τ_m , while the general dynamics remain the same over all periods. The factor $\sqrt{\tau_m}$ in the stochastic term in equation 13 contains a square root because of the scaling properties of Wiener processes (Steele 2001). In equation 13, parameter g has been set equal to 1 for identifiability.

3.3.1. Network Effects on Attribute Evolution. The effects discussed in the previous section are examples of exogenous effects. However, network-attribute coevolution studies usually focus on how the local network environments of actors and the characteristics of the actors to whom they are connected (i.e., their “alters”) affect their attribute dynamics. For example, the number of friends a student has or the self-esteem of his or her friends may affect that student’s self-esteem. We can model how actors are influenced by their alters in various ways, combining information on the current network state and the alters’ current attribute values.

Table 1 gives an overview of some network effects on attribute dynamics that have been implemented so far. For each of the effects, we give an example hypothesis in the context of the study discussed in Section 1. The actor’s network position itself may influence his or her behavior: For example, the indegree, outdegree, or reciprocated degree might lead to an increase or decrease in the value of the attribute.

Table 1. Network-Dependent Effects Currently Implemented for Modeling Continuous Attribute Dynamics

Effect	Effect Formula ^a	Illustration ^b	Example Hypothesis
Indegree	$\sum_j x_{ji} (= x_{i+})$		Being popular leads to higher self-esteem.
Indegree (sqrt)	$\sqrt{x_{i+}}$	(Same as above; differences between high indegrees are relatively less important than the same differences between low indegrees.)	
Outdegree	$\sum_j x_{ij} (= x_{i+})$		Having (nominating) many friends leads to higher self-esteem.
Outdegree (sqrt)	$\sqrt{x_{i+}}$	(Same as above; differences between high outdegrees are relatively less important than the same differences between low outdegrees.)	
Reciprocated degree	$\sum_j x_{ij} x_{ji}$		Having many close (reciprocated) friendship ties leads to higher self-esteem.
Nonreciprocated degree	$\sum_j x_{ij} (1 - x_{ji})$		Having many unreciprocated friendship ties leads to lower self-esteem.
In-isolate	$1 - \max_j \{x_{ji}\}$		Having nobody calling you a friend leads to lower self-esteem.
Minimum alter	$\min_j \{x_{ij}(z_j - \bar{z})\}$		The higher the minimum value of the self-esteem of one's friends, the larger the increase in one's self-esteem.
Average alter	$\sum_j x_{ij}(z_j - \bar{z})/x_{i+}$ (0 if $x_{i+} = 0$)		The higher the average value of the self-esteem of one's friends, the larger the increase in one's self-esteem.
Maximum alter	$\max_j \{x_{ij}(z_j - \bar{z})\}$		The higher the maximum value of the self-esteem of one's friends, the larger the increase in one's self-esteem.

^aTime dependence is omitted for brevity. \bar{z} denotes the observed mean of z .
^bDarker colors represent higher values of the attribute. Shaded area indicates the actors who exceed influence.

Degree-related effects can be considered in their raw form or in a transformation (e.g., Steglich et al. 2010). If the effect of an additional tie decreases with the number of ties an actor already has, this can be accounted for by a square root transformation. Especially for a right-skewed degree distribution, where some actors have a very high degree, such a transformation is natural.

Actors may be affected by the attribute values of their alters. Three remarks about social influence may be noted here. First, we use the term *social influence* generally for all ways in which the network combined with the behavior of other actors in the network affects the behavior of actors. This is more general than the narrower meaning of social influence that refers only to ways in which the behavior of the influencing actor becomes more similar to the behavior of the actor being influenced, and it is illustrated in the following for some of the effects. Second, while some theories of social influence are formulated in terms of one actor who is being influenced by one other actor, neglecting third and additional actors, because of the empirical goals in coevolution studies, we must necessarily keep in mind that, in most cases, actors are surrounded by several other actors. The behavior of the focal actor is potentially influenced by all actors who are tied to the focal actor, and an aggregation step is necessary from the level of ties to the level of the personal network. Third, we interpret the tie from i to j as an assertion that j is subjectively meaningful to i (cf. Lomi et al. 2011) and therefore accord a primary role to influence coming from those to whom the focal actor has an outgoing tie, relegating a minor importance to influence from those actors from whom the focal actor has an incoming tie.

The theoretical mechanisms behind social influence can vary by context. The choice to model influence by a particular effect is thus a theoretical one. Influence effects will have to reflect theories and potential mechanisms that could apply to the phenomenon being analyzed. Table 1 shows three examples of influence effects. As the corresponding hypotheses in the context of self-esteem could be artificial in some cases, we will discuss them instead for deviant behavior. The attribute-related influence effects in Table 1 are centered by the mean observed attribute value \bar{z} . Centering the attribute values in the effects gives meaning to the zero effect for actors without alters; this zero effect equals the effect for actors with average alters.

Having at least one friend who is not involved in deviant behavior may keep a person on the straight and narrow. In this case, a person's

score on a deviance scale could increase when his or her least deviant friend becomes more deviant (minimum alter effect). If the least deviant friend of individual A is less deviant than the least deviant friend of individual B and this results in a higher increase in deviance of B than of A, we can consider this a form of social influence.

An alternative could be that having at least one deviant friend has a large impact on someone's deviant behavior. In this case, a person's deviance score could increase when his or her most deviant friend becomes more deviant (maximum alter effect). This may lead a person to deviate from the norm in his or her friendship group. Social influence thus does not necessarily imply that actors become more similar to their friends over time. Note that the person who is the least or most deviant friend may change over time.

A third possibility is that a person is affected by the average deviance level of his or her friends (average alter effect). In this case, the positive effect of the nondeviant individuals and the negative effect of the deviant individuals are assumed to even each other out. The average alter effect is a common operationalization of social influence in studies using the stochastic actor-oriented model. The idea of using a (weighted) average alter effect to model influence goes back to classical sociological models (e.g., Abelson 1964; French 1956); see Flache et al. (2017) for a recent overview of formal models of social influence.

This list can be extended with many other effects. A large variety of these have already been defined for discrete attribute variables in the stochastic actor-oriented modeling framework (Ripley et al. 2018), and many of these allow for a straightforward generalization to the case of continuous attributes. For example, in the discrete attribute evolution model, the total in-alter effect is defined by actor i 's behavior multiplied by the sum of the behaviors of his or her in-alterers, $z_i(\sum_j x_{ji}z_j)$ (Ripley et al. 2018). In the continuous attribute evolution model, the total in-alter effect would be $\sum_j x_{ji}z_j$.

Note that the effects in the discrete model are interpreted as effects of the type of a utility, or negative potential, whereas in our model, it is of the type of a derivative. This is why the effects in the discrete model have the additional factor z_i . Disregarding the factor z_i is a strategy that turns many of the discrete attribute model effects into continuous attribute model effects. This principle works for effects in the discrete model defined by functions that are a multiple of z_i , which is the case for many but not all behavior effects. When input vector $u_i(t)$ includes functions

depending on z_i , equation 12 is no longer the solution to stochastic differential equation 11.

3.3.2. Discrete-Time Consequences. Stochastic differential equations describe how continuous variables may evolve over time. They express a rate of change. However, observations are usually made at discrete time points. The distribution of the continuous variables at a certain time point is fully determined by the stochastic differential equation and the initial conditions, yet for most models, it is impossible to derive an explicit expression for this distribution. Bergstrom (1984) addressed this problem for systems of linear stochastic differential equations that model the coevolution of multiple continuous variables. He showed that under certain conditions, discrete-time observations exactly satisfy a system of stochastic difference equations. His so-called exact discrete model links the discrete-time parameters with the continuous-time parameters.

Equation 11 is the one-dimensional case of the model addressed by Bergstrom (1984). For this model, the exact discrete model reduces to an expression very similar to what we have seen in Section 2 on stochastic differential equations (e.g., see Oud and Jansen 2000). Let $z_{i,t}$ denote the attribute value and $u_{i,t}$ the values of the effects in the input vector of actor i at time t . The exact discrete model states that after a time Δt , the value of the attribute of actor i is given by

$$z_{i,t+\Delta t} = A_{\Delta t} z_{i,t} + B_{\Delta t} u_{i,t} + w_{i,\Delta t}, \quad (14)$$

where $w_{i,\Delta t}$ can be considered the random error caused by the error process over Δt time, with a $\mathcal{N}(0, Q_{\Delta t})$ distribution, and where

$$A_{\Delta t} = e^{a\Delta t}, \quad B_{\Delta t} = \frac{1}{a}(e^{a\Delta t} - 1)b^\top, \quad Q_{\Delta t} = \frac{1}{2a}(e^{2a\Delta t} - 1)g^2. \quad (15)$$

For equation 13, modeling attribute dynamics based on more than two measurement moments, these coefficients are given by

$$A_{\Delta t} = e^{a\tau_m\Delta t}, \quad B_{\Delta t} = \frac{1}{a}(e^{a\tau_m\Delta t} - 1)b^\top, \quad Q_{\Delta t} = \frac{1}{2a}(e^{2a\tau_m\Delta t} - 1). \quad (16)$$

In the derivation of difference equation 14, it is assumed that the r effects in u_i are constant between t and $t + \Delta t$. In some cases, this assumption clearly does not hold—for example, when the average alter effect is included in the model. In Section 3.4, we reflect on the consequences of this approximation on the coevolution model.

3.4. Integration of Network and Attribute Model

The network-attribute coevolution model is specified by the rates λ_i defining the pace of the network change, the objective function (equation 10) modeling the mechanisms by which actors make network changes, and the exact discrete model (equation 14), corresponding to a stochastic differential equation. The stochastic differential equation models both the pace and the direction of change in the continuous actor attributes.

In the coevolution model, the network evolves in “jumps” of one tie change, while the actor attributes evolve gradually. The Markov chain for the network evolution is fully specified by its infinitesimal generator matrix or intensity matrix Q (Norris 1997), of which the entries $q(x, x')$ indicate the rate at which network state x changes into \tilde{x} ,

$$q(x, \tilde{x}) = \lim_{dt \rightarrow 0} \frac{P(X(t+dt) = \tilde{x} | X(t) = x)}{dt}. \quad (17)$$

The diagonal entries of Q are chosen such that the rows of Q sum to zero. The intensity matrix is given by

$$q(x, \tilde{x}) = \begin{cases} \lambda_i p(x^{(\pm ij)}) & \text{if } \tilde{x} = x^{(\pm ij)} \text{ for some } i \neq j \\ 0 & \text{if } \tilde{x} \neq x^{(\pm ij)} \text{ for some } i, j \\ -\sum_{x \neq \tilde{x}} q(x, \tilde{x}) & \text{if } \tilde{x} = x. \end{cases} \quad (18)$$

Note that the off-diagonal entries that are nonzero correspond to network changes that involve only a single tie change. The waiting time until a transition out of state x is the minimum of the exponentially distributed waiting times with rates $q(x, \tilde{x})$ to change from x to \tilde{x} , and thus exponentially distributed with rate $-q(x, x)$. Given the occurrence of an instantaneous transition out of x , the conditional probability that the state shifts from x to \tilde{x} is

$$\frac{q(x, \tilde{x})}{-q(x, x)} = \begin{cases} \frac{\lambda_i}{\lambda_+} p(x^{(\pm ij)}) & \text{if } \tilde{x} = x^{(\pm ij)} \text{ for some } i \neq j \\ 0 & \text{else.} \end{cases} \quad (19)$$

This transition probability decomposes into the probability λ_i/λ_+ that actor i gets the opportunity to make a network change and the probability $p(x^{(\pm ij)})$ that the change made by actor i consists of a tie change to alter j . Using the exact discrete model to evaluate how much the dynamic actor attributes have evolved between two consecutive tie

changes, we can thus set up a simulation scheme for the network-attribute coevolution process in period m , which consists of seven steps:

1. Initialize: set $t=0$, $x=x(t_m)$, $z=z(t_m)$ and compute $u_i=u_i(x, z)$ for all actors i .
2. Sample Δt from an exponential distribution with rate λ_+ .

While $t + \Delta t < 1$,

3. Update z_i to a sample from a $\mathcal{N}(A_{\Delta t} z_i + B_{\Delta t} u_i, Q_{\Delta t})$ distribution, for all actors i .
4. Select actor $i \in \{1, \dots, n\}$ according to probabilities λ_i/λ_+ .
5. Select alter $j \in \{1, \dots, n\}$ according to probabilities $p(x^{(\pm ij)})$.
6. Update: set $t=t+\Delta t$ and $x=x^{(\pm ij)}$ and compute $u_i=u_i(x, z)$ for all actors i .
7. Sample a new Δt from an exponential distribution with rate λ_+ .
Return to Step 3.

In the simulation scheme, there is a waiting time until a new network change is drawn (Steps 2 and 7), the actor attributes are updated (Step 3), the actor who will make a change is determined (Step 4), and the tie change is determined (Step 5). To reach $t=1$, the attributes of all actors are updated for a final time (Step 3). The choice for a simulation time length of 1 is arbitrary. The actual length $t_{m+1} - t_m$ of period m is captured in the rate parameters and the parameters of the stochastic differential equation. The definitions of $A_{\Delta t}$, $B_{\Delta t}$, and $Q_{\Delta t}$ in the aforementioned simulation scheme are as in equations 15 or 16, depending on the choice of differential equation.

For simulation purposes, we assume that u_i is constant between consecutive tie changes at times t and $t+\Delta t$. This is not always true. The network is constant between t and $t+\Delta t$, so any effects in u_i that are functions of only the network and individual and dyadic covariates are constant between t and $t+\Delta t$. However, if u_i contains an effect, such as the average alter effect, that depends on the attribute values z_j of other actors $j \neq i$ in the network, the assumption is no longer valid as the values z_j evolve between t and $t+\Delta t$.

If the Δt is small, the errors introduced by the approximation are small as well, as is shown by Niezink and Snijders (2017) in a simulation study. For a small Δt , the number of actors n and/or the rate

parameters λ_m need to be sufficiently large as $E(\Delta t) = 1/(n\lambda_m)$. Niezink and Snijders (2017) explore the effect of $E(\Delta t)$ on the approximation error in the values of the actor attributes. Given the results obtained in the simulation study, any deviation in parameter estimates is likely to be negligible when $n\lambda_m$ is larger than 100. This is true for most of the data sets studied in practice, which have the size of a school class ($n = 25$) or larger.¹

For very small networks with little change between observations, deviations in simulated attribute values will be larger. Splitting up the time interval $[t, t + \Delta t)$ between two consecutive network changes in a simulation into smaller parts and evaluating the exact discrete model for each smaller part would decrease the deviation.

4. EFFECT SIZES

An important difficulty in working with stochastic actor-oriented models is the interpretation of parameters in a nonstandardized fashion. The parameters in the standard stochastic actor-oriented model for discrete behavior variables are nonstandardized coefficients in the multinomial logit model for the discrete changes in network and behavior. Multinomial logit model parameters are difficult to interpret, and their arrangement in this complex network model adds to the difficulties in interpretation. The relation of the stochastic differential equation model to linear regression can be used to have a better understanding of the parameters, with analogues of the proportion of explained variance and effect sizes.

4.1. Parameter Interpretation

Estimated parameters in attribute dynamics models (equations 11 and 13) represent the strength of effects on change in attribute values. Using the exact discrete model (see again Section 3.3.2), we can assess the implication of this model for expected change trajectories. Considering these change trajectories helps in interpreting parameter sizes.

Assume, by way of example, that in a simple coevolution model, the stochastic differential equation for evolution of a mean-centered attribute variable was given by

$$dZ_i(t) = [aZ_i(t) + b_0 + b_1v_i - b_2X_{+i}(t)]dt + g dW_i(t), \quad (20)$$

where $X_{+i}(t)$ denotes the indegree of actor i at time t , representing popularity in a friendship network. This differential equation contains a feedback parameter a , an intercept parameter b_0 , a parameter b_1 for binary (0/1) covariate v_i , an indegree parameter b_2 , and a diffusion parameter g . The model illustrates the general model (equation 11). Suppose that this model was estimated as

$$dZ_i(t) = [-0.4Z_i(t) + 0.1 + 0.2v_i - 0.05X_{+i}(t)]dt + 0.4dW_i(t). \quad (21)$$

The parameters in equation 13, for more than two measurement moments, can be interpreted along the same lines as discussed in the following.

To simplify parameter interpretation, we approximate the indegree $X_{+i}(t)$ by a constant x_{+i} . All input variables $u_i(t)$ being constant, exact discrete model equation 14 shows that

$$\begin{aligned} E(Z_i(t)|z_i(0)) &= e^{at}z_i(0) + \frac{1}{a}(e^{at} - 1)(b_0 + b_1v_i + b_2x_{+i}) \\ &= e^{-0.4t}z_i(0) + \frac{1}{-0.4}(e^{-0.4t} - 1)(0.1 + 0.2v_i - 0.05x_{+i}) \\ &= 0.67^t z_i(0) + (1 - 0.67^t)(0.25 + 0.5v_i - 0.125x_{+i}), \end{aligned} \quad (22)$$

where t runs between 0 and 1. This expression represents the expected attribute trajectory for actor i between two measurement moments. The expected value is a weighted average of the initial score $z_i(0)$ and the theoretical equilibrium value $0.25 + 0.5v_i - 0.125x_{+i}$. The weights are e^{at} and $1 - e^{at}$. The variation about the expected value is

$$\begin{aligned} \text{var}(Z_i(t)|z_i(0)) &= \frac{1}{2a}(e^{2at} - 1)g^2 \\ &= \frac{1}{-0.8}(e^{-0.8t} - 1) \times 0.4^2 \\ &= 0.2(1 - 0.45^t). \end{aligned} \quad (23)$$

Figure 3 visualizes the attribute dynamics given in equation 21. Figure 3a shows the extent of the variation in 50 sample trajectories for one actor. Note that we assume the variation to be the same for all actors, much like the homoscedasticity assumption in a regression analysis. Figure 3b shows the effect of an actor's indegree on his or her attribute value. The differences between the trajectories for actors with different indegrees are small, especially considering the amount of random variation in Figure 3a.

As shown by the previous exposition, the continuous-time parameters are best interpreted in terms of their discrete-time consequences. Interpreting the value of 0.2 for the covariate parameter b_1 by itself is

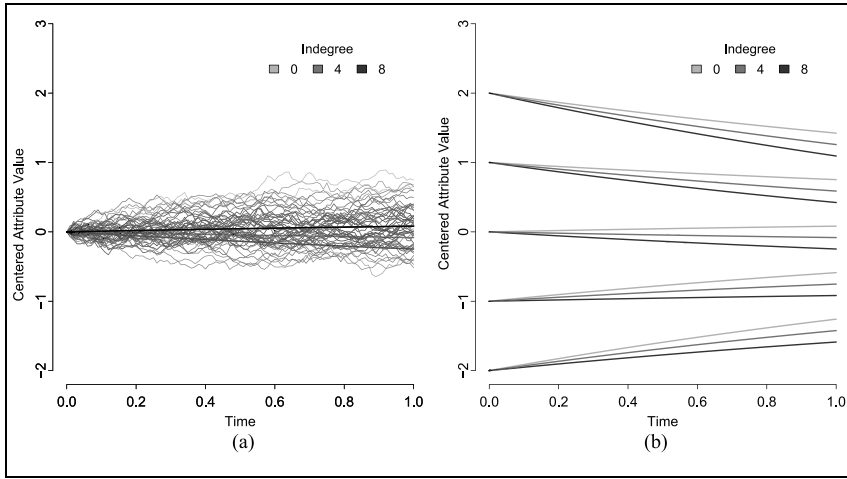


Figure 3. Visualization of the attribute dynamics given in equation 21.

Note. (a) The mean trajectory and 50 sample paths for an actor with an average initial attribute value, $v_i = 0$ and indegree 0. (b) The mean trajectories for actors with $v_i = 0$ for various initial attribute and indegree values.

difficult. The ratio $-b_1/a=0.5$ indicates the size of the difference between actors with $v_i=0$ and $v_i=1$ in the theoretical equilibrium. The equilibrium coefficients $-b_i/a$ are more fundamental entities than the b_i . This was discussed for a basic ordinary differential equation model by Nielsen and Rosenfeld (1981). The substantive interpretations of these coefficients can be given according to the same logic applied to coefficients in a linear regression model. Also, the equilibrium variance, $-g^2/(2a)=0.2$, is easier to interpret than the parameter g as it corresponds directly with the total intraindividual variance (Oravecz et al. 2011).

Nevertheless, the processes we study are often far from being in equilibrium, and focusing on the equilibrium coefficients may not be all-revealing. Therefore, we also evaluate the discrete-time consequence of the continuous-time model after one observation period. Since in the model the time between consecutive measurements is set to 1, the exact discrete model yields

$$\begin{aligned} E(Z_i(1)|z_i(0)) &= e^a z_i(0) + \frac{1}{a}(e^a - 1)(b_0 + b_1 v_i + b_2 x_{+i}) \\ &= 0.67 z_i(0) + 0.08 + 0.16 v_i - 0.04 x_{+i}, \end{aligned} \quad (24)$$

where the indegree is again taken to be constant. Moreover, $\text{var}(Z_i(1)|z_i(0))=0.11$, and the corresponding standard deviation is 0.33.

The coefficient $e^a = 0.67$ can be interpreted as representing the memory of the process, the dependence of $Z_i(1)$ on $z_i(0)$ (Nielsen and Rosenfeld 1981). The covariate coefficient 0.16 is much smaller than the equilibrium coefficient $-b_1/a = 0.5$ due to the weight $1 - e^a$. Each additional incoming tie decreases the expected change after one observation period of an actor's attribute value by 0.04. Though in reality $X_{+i}(t)$ is usually not constant, this line of reasoning provides a good approximation of the effect of the indegree on attribute change.

Filling out observation interval length $t = 1$ in equations 22 and 23 facilitates interpretation. Oud et al. (2012) illustrate this, presenting the discrete-time parameters corresponding to their observation interval of one year alongside continuous-time parameter estimates. The importance of also reporting the continuous-time parameters is stressed by examples given by, for example, Oud and Delsing (2010) and Voelkle et al. (2012). The continuous-time parameters are necessary to compare parameters between studies with different time intervals between measurements because they are not time-dependent. For example, if in Study 1 the time between two measurements was six months and in Study 2 this interval was one year, we need to fill out $t = 1$ for Study 1 and $t = 0.5$ for Study 2 to make a comparison possible.

4.2. Explained Variance

For stochastic actor-oriented models for network evolution, Snijders (2004) proposed a measure of explained variation using information theory. The measure is based on the entropy (Shannon 1984) in the probabilities involved in actors' network change decisions, where a low entropy indicates a high degree of certainty in the outcomes of network changes and thus high explained variation. Unfortunately, the measure is not easily interpretable and is therefore hardly applied.

In linear regression, the proportion of explained variance, usually denoted by R^2 , is a well-interpretable and widely used measure of "fit." As the discrete-time outcomes of our stochastic differential equation model for attribute dynamics can be considered in a regression framework, we can define an equivalent measure of the proportion of explained variance based on

$$\text{var}(Z_i(t_{m+1})|y(t_m)), \quad (25)$$

where $y(t_m)$ denotes the network and attribute state at observation moment t_m .

For a linear regression model $z_i = X_i^\top \beta + \epsilon_i$, $i = 1, \dots, n$, the proportion of explained variance is given by

$$R^2 = \frac{\text{var}(z_i) - \text{var}(\epsilon_i)}{\text{var}(z_i)} = 1 - \frac{\text{var}(\epsilon_i)}{\text{var}(z_i)}. \quad (26)$$

We can consider R^2 as a proportional reduction in the unexplained variance or equivalently, the proportional reduction in the mean squared error of prediction. If no covariates are available (a null or intercept-only model), the best predictor for z_i is $\mu = E(z_i)$, and the mean squared prediction error is $\text{var}(z_i) = E((z_i - \mu)^2)$. For the regression model with covariates, the best predictor for z_i is the regression value $E(z_i | X_i) = X_i^\top \beta$. The observed residual ϵ_i is the difference between the observed value z_i and this best predictor, and the mean squared prediction error under the regression model is $\text{var}(\epsilon_i)$. Equation 26 thus gives the proportional reduction in prediction error due to the inclusion of covariates.

The same ideas can be applied to define a proportion of explained variance in the context of the continuous attribute model presented here. For the stochastic differential equation, the null model M_0 contains a feedback parameter a , an intercept parameter b , and a diffusion or scale parameter. The best predictor for $Z_i(t_{m+1})$ under this model follows from the exact discrete model and is given by

$$E_{M_0}(Z_i(t_2) | y(t_1)) = e^a z_i(t_1) + (1 - e^a) \frac{b}{-a} \quad (27)$$

for model equation 11 and

$$E_{M_0}(Z_i(t_{m+1}) | y(t_m)) = e^{a\tau_m} z_i(t_m) + (1 - e^{a\tau_m}) \frac{b}{-a} \quad (28)$$

for model equation 13. This quantity can be computed using estimates of a and b (and for equation 28 also τ_m) based on all attribute data, without the need for simulations. The best predictor under a more elaborate model M ,

$$E_M(Z_i(t_{m+1}) | y(t_m)), \quad (29)$$

can be estimated based on simulations of the coevolution model under study. In line with equation 26, the proportional reduction in

unexplained variance, or the proportional reduction in prediction error for period m , is given by

$$R_m^2 = 1 - \frac{\sum_{i=1}^n (z_i(t_{m+1}) - E_M(Z_i(t_{m+1})|y(t_m)))^2}{\sum_{i=1}^n (z_i(t_{m+1}) - E_{M_0}(Z_i(t_{m+1})|y(t_m)))^2}. \quad (30)$$

Note that if the attribute dynamics do not depend on network characteristics, we can estimate the attribute model straightforwardly using the exact discrete model and likelihood maximization and determine the explained variance as in a standard regression model. This is also possible if the network is constant and the model contains only purely structural and covariate effects. Note also that while R^2 is always positive for linear regression models, for other types of models such as nonlinear regression or multilevel models, negative values are known to occur as well.

5. THE COEVOLUTION OF FRIENDSHIP, ALCOHOL USE, AND SELF-ESTEEM

This section investigates the interplay of friendship dynamics and the dynamics of alcohol use and self-esteem among adolescents. We use a stochastic actor-oriented model to study the coevolution of a network (friendship), a discrete actor variable (alcohol use), and a continuous actor variable (self-esteem). Steglich et al. (2010) advocated the use of the stochastic actor-oriented model to distinguish peer selection from social influence, two social mechanisms leading to network autocorrelation. Since then, many researchers have followed in disentangling selection and influence in various contexts. Nevertheless, few studies have considered the conditions under which selection and influence occur. Some actors may be more susceptible to these processes than others. Schaefer (2016) studied whether adolescents with particular risk factors, such as having low self-control or weak attachments to protective institutions (e.g., family or school), have a greater risk of befriending substance-using peers, who could later become a source of negative influence. However, not all adolescents may be equally susceptible to influence. Discovering the characteristics of the adolescents who are most susceptible to influence of their peers is important. Compared with trying to change a person's friendship ties, interventions aimed at

individual characteristics are easier to implement (e.g., in a personal skills training context) and often more ethical.

In this study, we consider self-esteem as a potential buffer for the effect of peers on the behavior of adolescents. Adolescents with high self-esteem may be less susceptible to influence than their low self-esteem peers. We reanalyze the data studied by Steglich et al. (2010), considering the role of self-esteem in friendship and alcohol use dynamics. Furthermore, we assess the effects of popularity and alcohol use on self-esteem.

5.1. *Data*

The data are part of the Teenage Friends and Lifestyle Study (Pearson and Mitchell 2000; Pearson and West 2003), which aimed to identify the mechanisms by which attitudes toward smoking and smoking behavior itself change during early to midadolescence. Students in a cohort at a secondary school in Glasgow were followed over a two-year period (February 1995–January 1997). Of the 160 students in the cohort, aged 12 to 13 at the beginning of the study, 150, 146, and 137 participated at the first, second, and third measurement, respectively. We include all 160 students in the analysis, taking into account the changes in composition. The students were asked to nominate up to six friends from their cohort and answer questions about various behaviors and attitudes, including social relations and risk behavior. Previous studies of these data have addressed the coevolution of friendship and taste in music (Steglich, Snijders, and West 2006) and that of friendship and cannabis use (Pearson, Steglich, and Snijders 2006).

Alcohol consumption frequency was measured on a scale ranging from 1 (not at all) to 5 (more than once a week). Self-esteem was measured by a 10-item scale, based on Rosenberg (1941), with items such as “I am easy to like” and “I often wish I was someone else.” The items were measured on a scale ranging from 0 (strongly agree) to 3 (strongly disagree). The self-esteem score was calculated as the average over all items after reverse coding the negatively formulated items so that a high score corresponds to high self-esteem. Students also reported their sex (0 = male, 1 = female).

For descriptive statistics of the friendship network and alcohol use data, we refer to Steglich et al. (2010). The distribution of the self-esteem data at the three measurements is shown in Figure 4, where a

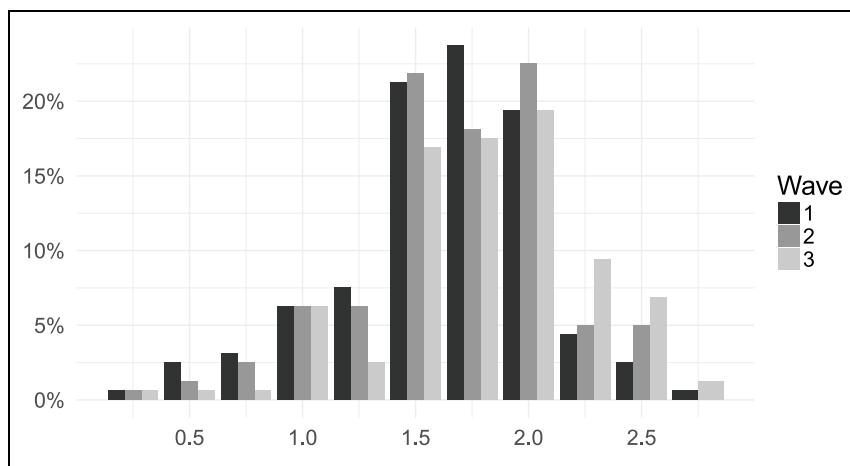


Figure 4. Observed distribution of self-esteem at the three measurements.

slight increase of self-esteem can be seen over time. The average self-esteem score increases from 1.63 at the first measurement to 1.68 at the second and 1.79 at the third. For the analysis, self-esteem is centered by subtracting the overall mean of 1.70.

5.2. Model

In order to study the effect of self-esteem on students' susceptibility to peer influence on drinking behavior, we need to model self-esteem as a coevolving dependent variable. We thus study the coevolution of friendship, alcohol use, and self-esteem, modeling the evolution of a network, a discrete actor variable, and a continuous actor variable simultaneously. Two models are estimated. In the first model, the "evolution model," we study the dynamics of friendship, alcohol use, and self-esteem separately. Formally, this can be done in one joint model, where the three dependent variables are specified as being mutually independent. The interplay among the three dependent variables is studied in the second model, the "coevolution model." In this model, we assess through an interaction term whether students' susceptibility to peer influence on alcohol use depends on their level of self-esteem. We estimate parameters by the methods of moments. See Appendix A and Appendix B for a discussion of parameter estimation and standard error estimation.

Missing data are imputed for simulation purposes but disregarded in the computation of the statistics of the moment equations (Huisman and Steglich 2008; Ripley et al. 2018).

5.2.1. Friendship Dynamics. In the friendship dynamics part of the model, we first include structural effects in the objective function. The outdegree effect represents the balance between creating and dropping ties, and it is like an intercept. We also model the tendency to reciprocate friendship nominations (reciprocity) and the tendency for actors to befriend the friends of their friends (transitivity) and the interaction of these two effects. Reciprocity and transitivity usually play an important role in friendship dynamics, but their effect is mostly not additive, resulting in a negative interaction effect. As explained by Block (2015), the tendency toward reciprocation of friendships within transitive groups is usually lower than it is outside of transitive groups.

Apart from the basic outdegree effect, we include three other degree-related effects: the effect of current popularity (number of incoming ties, indegree) on receiving friendship nominations (indegree popularity) and sending friendship nominations (indegree activity) and the effect of current network activity (number of outgoing ties, outdegree) on nominating friends (outdegree activity).

We also assess the effects of alcohol use and self-esteem on friendship dynamics by including their ego effects and alter effect in the network objective function. These effects measure the differential tendency of students with higher values to nominate friends and receive friendship nominations, respectively. We model the differential attractiveness of students with high self-esteem to other students with high self-esteem by an interaction effect of ego's and alter's self-esteem scores. We also include the interaction of ego's and alter's alcohol use. Finally, we account for effects of gender (ego, alter, same) on the dynamics of the friendship network.

5.2.2. Self-esteem Dynamics. In the coevolution model, we also include, apart from the feedback, intercept and scale parameters, the effect of alcohol use on change in self-esteem and that of popularity, as measured by a student's indegree in the friendship network. The stochastic differential equation for the self-esteem dynamics in period $m = 1, 2$ is thus given by

$$dZ_i(t) = \tau_m [a Z_i(t) + b_0 + b_1 A l_i(t) + b_2 X_{+i}(t)] dt + \sqrt{\tau_m} dW_i(t), \quad (31)$$

where $Al_i(t)$ denotes the alcohol use score of actor i at time t .

Since we analyze the centered esteem scores instead of the original ones, we can meaningfully interpret the intercept parameter b_0 . When no other input effects are included in the model ($b_1 = b_2 = 0$), the intercept represents the general tendency of students to decrease or increase in self-esteem.

5.2.3. Alcohol Use Dynamics. The alcohol use dynamics, like the friendship dynamics, are modeled in the discrete Markov chain framework. The base components of the alcohol use objective function are the linear and quadratic shape effects, which capture the basic shape of the alcohol use distribution. The model contains one main friendship-related peer influence component: the effect of the average alcohol use among friends on adolescent alcohol use (average alter effect). As we are interested in whether self-esteem moderates the strength of peer influence on alcohol use, we include the interaction effect of ego's self-esteem and the average alter effect on alcohol use. We also take into account the potential direct effect of self-esteem on alcohol use.

5.3. Results

The results for the evolution model and the coevolution model are shown in Table 2. Based on 1,000 simulations of the coevolution model and using the procedure developed by Lospinoso (2012) implemented in the RSiena package (Ripley et al. 2018), we can conclude that the model fits the network data well in terms of its outdegree distribution ($p = .10$), indegree distribution ($p = .60$), and triad census ($p = .09$). The alcohol and self-esteem data also fit fairly well, as shown in Figure 5. In the following discussion, we first explain the friendship network dynamics results and then the results for the alcohol use and self-esteem dynamics.

We find that students with higher self-esteem tend to send more friendship nominations (self-esteem ego). Apart from this, students' self-esteem seems to have little effect on the evolution of friendship ties, and the same holds for their alcohol use.

The estimates of the purely structural effects do not change much when the alcohol and self-esteem effects are included in the model. Students have a tendency to reciprocate friendship ties and prefer relationships with their friends' friends (positive transitive triplets), but these effects are not additive (negative transitive reciprocated triplets).

Table 2. Coevolution of Friendship, Alcohol Use, and Self-esteem

	Evolution Model		Coevolution Model	
	Estimate	(SE)	Estimate	(SE)
Friendship dynamics				
Rate period 1	13.17 ^a	(1.24)	13.20 ^a	(1.36)
Rate period 2	10.77 ^a	(1.12)	10.88 ^a	(1.08)
Outdegree (density)	-2.98 ^a	(.21)	-2.92 ^a	(.20)
Reciprocity	2.62**	(.15)	2.58**	(.15)
Transitive triplets	.88**	(.05)	.88**	(.06)
Transitive reciprocated triplets	-.54**	(.08)	-.55**	(.09)
Indegree popularity	-.022	(.020)	-.022	(.019)
Indegree activity	-.16**	(.05)	-.12*	(.04)
Outdegree activity	.054	(.029)	.025	(.031)
Female ego	-.002	(.124)	.049	(.10)
Female alter	-.10	(.12)	-.13	(.09)
Same sex	.70**	(.10)	.68**	(.09)
Alcohol ego			.086	(.074)
Alcohol alter			-.091	(.058)
Alcohol ego × alter			.12	(.06)
Self-esteem ego			.39*	(.15)
Self-esteem alter			-.17	(.14)
Self-esteem ego × alter			-.10	(.26)
Alcohol use dynamics				
Rate period 1	1.33 ^a	(.26)	1.52 ^a	(.30)
Rate period 2	2.40 ^a	(.51)	2.58 ^a	(.58)
Linear shape	.42 ^a	(.10)	.45 ^a	(.14)
Quadratic shape	-.28 ^a	(.08)	-.61 ^a	(.24)
Average alter			1.32*	(.66)
Self-esteem			-.14	(.32)
Self-esteem × average alter			-.29	(.88)
Self-esteem dynamics				
Scale period 1	.12 ^a	(.01)	.12 ^a	(.02)
Scale period 2	.17 ^a	(.02)	.17 ^a	(.01)
Feedback	-2.41 ^a	(.42)	-2.49 ^a	(.43)
Intercept	.52*	(.19)	.63	(.54)
Alcohol			-.22	(.30)
Friendship indegree			-.029	(.126)

^aNot tested.

* $p < .05$. ** $p < .001$.

Moreover, students who are mentioned by many others as a friend have a lower tendency to nominate others as friends (negative indegree activity). Finally, we find strong evidence of homophily based on sex (positive same gender).

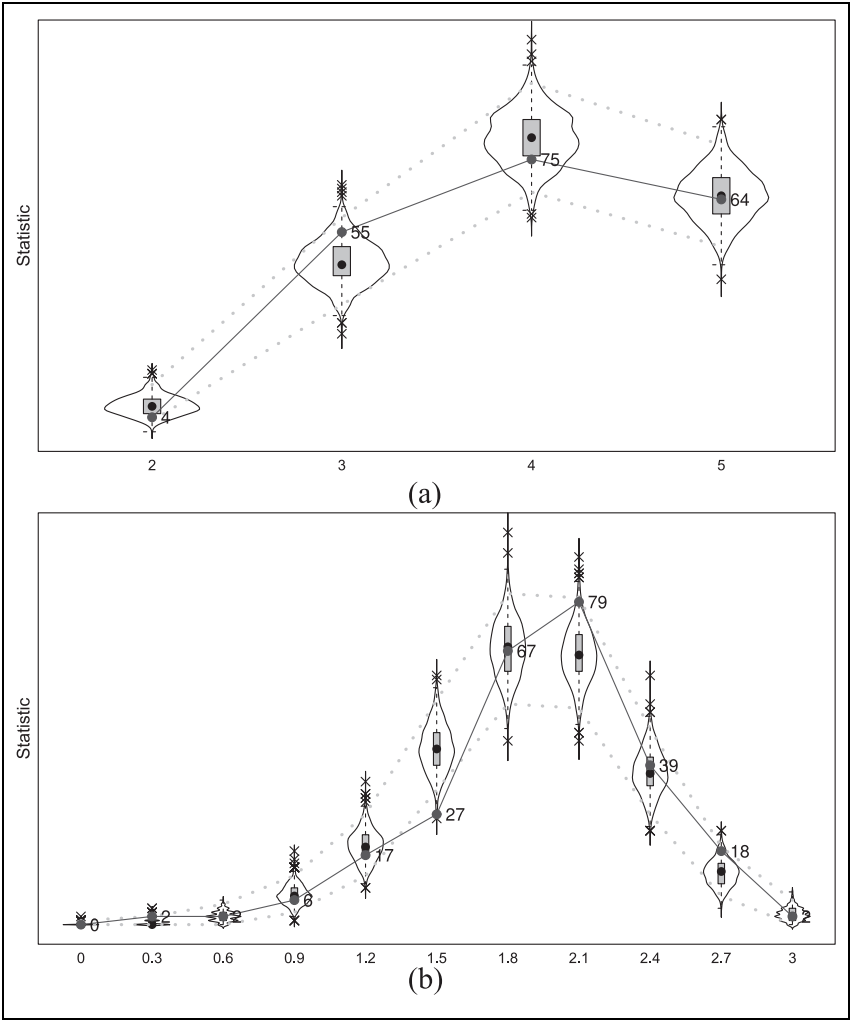


Figure 5. Goodness-of-fit plots for alcohol use and self-esteem distribution based on 1,000 data sets simulated with the coevolution model parameters (see Table 2).

Note. (a) Alcohol use distribution, $p = .47$. (b) Self-esteem distribution, $p = .07$. The numbers and solid lines represent the values observed at the end of periods 1 and 2.

Peer influence is the social mechanism of key interest in our model of how students' alcohol use changes over time. Table 2 shows that friends indeed have an effect on students' alcohol intake (positive

average alter). However, we find no evidence that susceptibility to peer influence differs by students' self-esteem level (self-esteem \times average alter). Students' self-esteem does not appear to directly affect their alcohol use either. The results for the full model without the self-esteem and average similarity interaction effect—not presented here—are comparable to those for the coevolution model. In this model, the average alter effect is 1.34 ($SE = 0.60$), and the effect of self-esteem on alcohol use is -0.13 ($SE = 0.33$).

Considering the self-esteem dynamics model results for the evolution model, we find that self-esteem significantly increases over the course of the study period (positive intercept). This corresponds to the distributions shown in Figure 4. The sizes of the intercept and feedback parameter and the scale parameter for period 2 can be interpreted by considering again Figure 3a. The parameters in the estimated differential equation

$$\begin{aligned} dZ_i(t) &= 0.17[-2.41 Z_i(t) + 0.52]dt + \sqrt{0.17} dW_i(t) \\ &= [-0.41 Z_i(t) + 0.09]dt + 0.41 dW_i(t) \end{aligned} \quad (32)$$

are very similar to those used for simulating the trajectories in Figure 3a. The figure depicts the expected trajectory and gives an idea about the amount of uncertainty on the trajectory for a student with a self-esteem score of 1.70 (the overall observed average) at the beginning of period 2. The size of the random fluctuations is similar in the coevolution model and quite large compared with the effects of alcohol use or popularity on self-esteem. To see this, compare the parameter sizes of alcohol use and self-esteem in the estimated coevolution model

$$\begin{aligned} dZ_i(t) &= 0.17[-2.49 Z_i(t) + 0.63 - 0.22 A l_i(t) - 0.03 X_{+i}(t)]dt + \sqrt{0.17} dW_i(t) \\ &= [-0.42 Z_i(t) + 0.11 - 0.04 A l_i(t) - 0.005 X_{+i}(t)]dt + 0.41 dW_i(t) \end{aligned} \quad (33)$$

for period 2 with the size of the popularity effect (-0.04) in equation 24, which was used to generate Figure 3b. This figure showed that the differences between trajectories for actors with different indegrees are small given the amount of random variation. In our estimated model (equation 33), the popularity effect (-0.005) is approximately 10 times smaller! The effect of alcohol use on self-esteem seems to be larger, but note that the range of the alcohol scale is 4 while the indegrees in Figure 3b vary from 0 to 8.

Table 2 also shows that there is no significant effect of popularity or alcohol use on self-esteem in the coevolution model. Moreover, popularity and alcohol use do not improve the model in terms of explained variance for self-esteem. Based on 1,000 Monte Carlo simulations of the coevolution process, we find that for periods 1 and 2 the explained variance estimates are $R_1^2 = 0.002$ and $R_2^2 = -0.006$, respectively. The negative value of R_2^2 for period 2 means that the self-esteem model that includes popularity and alcohol as predictors fits the data worse than the null model.

6. DISCUSSION

This article presented a model for studying the coevolution of social networks and continuous attributes of network actors. The model extends the stochastic actor-oriented model for network evolution (Snijders 2001). The model discussed by Snijders et al. (2007) and Steglich et al. (2010) requires continuous attributes to be discretized, but in the model presented here, this is no longer necessary. Continuous attributes arise as representations not only for variables that are “really” continuous, such as income or length or weight dimensions, but also as results of multi-item scales such as psychological constructs, approximations to counts that have a wide range, and measured variables that are aggregations of many decisions and circumstances such as performance of individuals or companies.

We illustrated the proposed method with a study of the dynamics of friendship, alcohol use, and self-esteem among adolescents. This study advocates the elaboration of the “selection versus influence” narrative by considering under which circumstances peer selection based on shared characteristics and social influence occur. In the study, we consider whether the susceptibility of students to peer influence on alcohol use differs according to their self-esteem, but we do not find evidence for such differential susceptibility. This example was presented not so much because of its substantive results—the sample size is on the low side for such a question—but as an illustration of the type of research question for which this method could be used.

The continuous attribute dynamics are modeled by a linear stochastic differential equation. We showed how given estimated parameters, formulas and figures can be of help in understanding continuous actor attribute dynamics and in the communication of results. Moreover,

stochastic differential equations give us access not only to average trajectories but also to information about the variability in these trajectories. Comparable information for discrete dynamic actor attributes in the stochastic actor-oriented model is not obtained in as straightforward an approach.

The linear stochastic differential equation model closely resembles (and in some cases is equivalent to) an ordinary linear regression model. Many generalizations of the ordinary regression model can be implemented for the stochastic differential equation as well. Examples include random effect and latent variable models. In this article, we assumed that the network represents one group. A study of coevolution processes in a sample of networks (e.g., Knecht et al. 2011) would require a multi-level extension of the method proposed here.

The resemblance of the attribute evolution model with the ordinary linear regression model at the same time instills awareness of potential modeling challenges. Questions about the validity of the linearity and homoscedasticity assumption follow naturally. Moreover, although discretization of continuous variables is no longer necessary due to the model extension, transformation might be. This transformation could be aimed at improving the validity of the distributional assumptions of the model, but it could also have a substantive objective. For example, the perception of the importance of a one-point change in an attribute value may differ in different ranges of the attribute spectrum. We can study transformations of continuous actor attributes to ensure that the assumption of an interval scale is reasonable.

While the network and discrete behavior coevolution model (Snijders et al. 2007) assumed that the behavior variable has a lower and upper bound, no such assumption was made for the continuous behavior model presented in this article. If a continuous variable is studied that takes values only in a certain range, simulated behavior trajectories will mostly lie in this range but also partly outside it. A stochastic differential equation model with reflecting boundary conditions could be developed to counter this.

The choice between a coevolution model with a continuous versus a discrete attribute is not merely a methodological issue or a matter of data, but it depends on the research question under study. For example, even in the case when the exact amount of alcohol consumed by students in a high school is known, drinking behavior is best modeled on a binary scale when the research is about peer influence on the onset of

drinking. An onset model (Greenan 2015) in the stochastic actor-oriented modeling framework would in this case match the research question better than a model with an ordinal or continuous drinking variable. The differences in results and model properties between the models for continuous attributes and discretized continuous attributes require further exploration.

APPENDIX A

Parameter Estimation

Stochastic actor-oriented models are generally too complicated for likelihoods or estimators to be written in a closed-form expression, which makes maximum likelihood estimation and Bayesian estimation complex. Although methods for maximum likelihood estimation (Snijders, Koskinen, and Schweinberger 2010) and Bayesian estimation (Koskinen and Snijders 2007) have been developed for models for discrete dependent attribute variables, the most straightforward way to estimate the model parameters is by a method of moments procedure, which is computationally less intensive. It is described in detail by Snijders (2001) and Snijders, Steglich, and Schweinberger (2007) and can be summarized as follows.

For each parameter θ_k in the model, a statistic S_k is selected that captures the variability in the data accounted for by this parameter. According to the method of moments (e.g., Bowman and Shenton 1985), parameter estimates are the values for which the expected data given the parameters and the observed data are most similar. Formally, the method of moments estimator $\hat{\theta}$ is the value of $\theta = (\theta_k)$ for which

$$E_{\hat{\theta}}\{S\} = s, \quad (A1)$$

where $S = (S_k)$ denotes the vector of statistics and s the observed outcome. This expression is referred to as the moment equation. In the context of the network-attribute coevolution model, parameters are estimated from panel data. In the moment equation, we can therefore condition on the observed initial state $y(t_m)$ of period m . This amounts to not modeling the initial state and thus making no assumptions about it. The parameter estimates $\hat{\theta}$ are defined as the solution to the conditional moment equation, given by

$$E_{\hat{\theta}}\{S_k(Y(t_m), Y(t_{m+1}))|Y(t_m)=y(t_m)\} = S_k(y(t_m), y(t_{m+1})) \quad (\text{A2})$$

for parameters specific to a period m , such as rates λ_m in the network model and scale parameters τ_m in the continuous attribute model, and

$$\sum_{m=1}^{M-1} E_{\hat{\theta}}\{S_k(Y(t_m), Y(t_{m+1}))|Y(t_m)=y(t_m)\} = \sum_{m=1}^{M-1} S_k(y(t_m), y(t_{m+1})) \quad (\text{A3})$$

for parameters that are assumed to be constant across the periods. The conditional expectations in equations A2 and A3 cannot be calculated explicitly, except for some trivially simple models. Therefore, parameter estimates are obtained by a stochastic iterative procedure, which is based on the Robbins-Monro (Robbins and Monroe 1951) algorithm and elaborated by Snijders (2001; 2017a). This procedure exploits the property that stochastic actor-oriented models can be used to simulate a coevolution process. Given an initial state $y(t_m)$ and parameters θ , the state $Y(t_{m+1})$ can be simulated and the conditional expectations approximated.

A.1. Statistics for the Conditional Moment Equation. For each of the parameters in the stochastic actor-oriented model, we need to select an appropriate statistic for the conditional moment equation. For the parameters in stochastic differential equation 11, the attribute model for one period of data, we propose the statistics

$$\text{feedback } a \quad \sum_i Z_i(t_2)z_i(t_1), \quad (\text{A4})$$

$$\text{attribute effect } b_k \quad \sum_i Z_i(t_2)u_{ik}(t_1), \quad (\text{A5})$$

$$\text{diffusion } g \quad \sum_i (Z_i(t_2) - z_i(t_1))^2. \quad (\text{A6})$$

These statistics are derived from an autoregression model that is closely related to differential equation 11 (Niezink and Snijders 2017). In case effects $u_i(t)$ are constant over the period of analysis, the statistics are the sufficient statistics for equation 11—that is, no other statistic can be calculated from the same observed data that provides additional information about the values of the parameters. In this particular situation, the method of moments and the maximum likelihood estimators for

these parameters are equal. For the parameters in the network part of the model, Snijders (2001) proposed the statistics

$$\text{network rate } \lambda_1 \quad \sum_{i,j} |X_{ij}(t_2) - x_{ij}(t_1)|, \quad (\text{A7})$$

$$\text{network effect } \beta_k \quad \sum_i s_{ik}(X(t_2), y(t_1)). \quad (\text{A8})$$

When the model is estimated based on data from more than two measurements, the statistics for the feedback, attribute effect, and network effect parameters are summed over all periods, as in equation A3. The diffusion statistic evaluated by period yields the statistic for the scale parameters τ_m in the model in equation 13. Also the network rate statistic is evaluated by period. For a discussion of how cross-lagged statistics are used in equations A5 and A8 to disentangle selection and influence, we refer to Snijders et al. (2007) and Niezink and Snijders (2017).

APPENDIX B

Standard Error Estimation

The standard errors of $\hat{\theta}$ are obtained as the square roots of the diagonal elements of the approximate covariance matrix

$$\text{cov}(\hat{\theta}) \approx D_{\hat{\theta}}^{-1} \Sigma_{\hat{\theta}} (D_{\hat{\theta}}^{-1}) \quad (\text{B1})$$

(Bowman and Shenton 1985). Here $D_{\hat{\theta}}$ denotes the Jacobian matrix of partial derivatives of the statistics S with respect to the parameters θ and $\Sigma_{\hat{\theta}}$ the covariance matrix of the statistics. Matrices $D_{\hat{\theta}}$ and $\Sigma_{\hat{\theta}}$ are evaluated at the estimate $\hat{\theta}$ through simulations.

Neither the Jacobian matrix $D_{\hat{\theta}}$ nor the covariance matrix of the statistics $\Sigma_{\hat{\theta}}$ can be expressed in closed form. Therefore, they are determined using Monte Carlo estimation—that is, based on a large number of data sets simulated under the estimated model with estimated parameters $\hat{\theta}$. The estimate of Σ is simply the sample covariance matrix of the values of the statistics S in the simulated data sets.

For the estimation of the Jacobian D , Schweinberger and Snijders (2007) proposed an estimator based on the idea that the Jacobian can be rewritten as

$$D_{\theta} = \frac{\partial S}{\partial \theta} = E_{\theta} \left(S \frac{\partial \log p(Y)}{\partial \theta'} \right). \quad (\text{B2})$$

Here, Y denotes the complete data corresponding to a stochastic actor-oriented model analysis. It consists of the observed data, the holding times of the Markov process, and the sequence of all changes in the modeled time interval. The complete data likelihood is denoted by $p(Y)$. The complete data scores can be computed using the expressions derived by Schweinberger and Snijders (2007) and Niezink and Snijders (2017).

Based on N Monte Carlo simulations of the data, Y_1, \dots, Y_N , given the moment estimator $\hat{\theta}$, we can estimate expression B2 by

$$\frac{1}{N} \sum_{i=1}^N ((S_i - c)J_i), \quad (\text{B3})$$

where S_i denotes the statistics evaluated on simulated data Y_i , J_i denotes the complete data score $\partial \log p(Y_i) / \partial \theta'$, and c is a constant to reduce the Monte Carlo variance; note that $E(J_i) = 0$. In practice, we use $c = \frac{1}{N} \sum_{j=1}^N S_j$, which is not constant but is almost so (Schweinberger and Snijders 2007). This estimator applies variance reduction based on control variates and is currently the default for estimating standard errors in the stochastic actor-oriented model (Snijders 2017b). When parameters are estimated based on more than one period, the statistic S is the sum over statistics S^m computed separately based on the periods m between measurements at time t_m and t_{m+1} . In this case, the Jacobian is estimated by

$$\frac{1}{N} \sum_{i=1}^N \left(\sum_{m=1}^{M-1} (S_i^m - c_m) J_i^m \right), \quad (\text{B4})$$

where J^m is the complete data score and c_m the constant to reduce Monte Carlo variance based on the data in period m .

Acknowledgments

We gratefully acknowledge the editor and the anonymous reviewers as well as Christian Steglich for helpful comments and suggestions.

Funding

The first author was funded by the Netherlands Organization for Scientific Research (NWO) under grant 406-12-165.

Note

1. A high attribute rate of change relative to the network rate of change will likely result in a larger approximation error than a relatively low attribute change rate. However, as the attribute change in the simulation study by Niezink and Snijders (2017) is already rather extreme compared with what is usual in observations, the rule of thumb formulated here will apply to most situations.

References

- Abelson, Robert P. 1964. "Mathematical Models of the Distribution of Attitudes under Controversy." Pp. 142–60 in *Contributions to Mathematical Psychology*, edited by N. Frederiksen and H. Gulliksen. New York: Holt, Rinehart, and Winston.
- Agneessens, Filip, and Rafael Wittek. 2008. "Social Capital and Employee Well-being: Disentangling Intrapersonal and Interpersonal Selection and Influence Mechanisms." *Revue Française de Sociologie* 49:613–37.
- Almquist, Zack W., and Carter T. Butts. 2014. "Logistic Network Regression for Scalable Analysis of Networks with Joint Edge/Vertex Dynamics." Pp. 273–321 in *Sociological Methodology*. Vol. 44, edited by T. F. Liao. Thousand Oaks, CA: Sage Publications.
- An, Weihua. 2014. "Models and Methods to Identify Peer Effects." Pp. 514–32 in *The SAGE Handbook of Social Network Analysis*, edited by J. Scott and P. J. Carrington. London: SAGE Publications.
- Beltrami, Edward. 1993. *Mathematical Models in the Social and Biological Sciences*. Boston: Jones and Bartlett.
- Bergstrom, Albert R. 1984. "Continuous Time Stochastic Models and Issues of Aggregation Over Time." Pp. 1146–212 in *Handbook of Econometrics*. Vol. 2, edited by Z. Griliches and Michael D. Intriligator. Amsterdam, Netherlands: Elsevier Science Publishers.
- Blalock, Hubert M. 1969. *Theory Construction: From Verbal to Mathematical Formulation*. Englewood Cliffs, NJ: Prentice-Hall.
- Block, Per. 2015. "Reciprocity, Transitivity, and the Mysterious Three-cycle." *Social Networks* 40:163–73.
- Block, Per, Johan Koskinen, James Hollway, Christian Steglich, and Christoph Stadtfeld. 2018. "Change We Can Believe in: Comparing Longitudinal Network Models on Consistency, Interpretability, and Predictive Power." *Social Networks* 52: 180–91.
- Bowman, K. O., and L. R. Shenton. 1985. "Method of Moments." Pp. 467–73 in *Encyclopedia of Statistical Sciences*. New York: Wiley.

- Checkley, Matthew, Christian Steglich, Duncan Angwin, and Richard Endersby. 2014. "Firm Performance and the Evolution of Cooperative Interfirm Networks: UK Venture Capital Syndication." *Strategic Change* 23:107–18.
- Coleman, James S. 1964. *Introduction to Mathematical Sociology*. New York: Free Press of Glencoe.
- Coleman, James S. 1968. "The Mathematical Study of Change." Pp. 428–78 in *Methodology in Social Research*, edited by H. M. Blalock and A. B. Blalock. New York: McGraw-Hill.
- Cressie, Noel A. C., ed. 1991. *Statistics for Spatial Data*. New York: Wiley.
- Deboeck, Pascal R. 2012. "Dynamical Systems and Models of Continuous Time." Pp. 411–31 in *The Oxford Handbook of Quantitative Methods*. Vol. 2, edited by T. D. Little. New York: Oxford University Press.
- Dijkstra, Jan Kornelis, Scott D. Gest, Siegwart M. Lindenberg, René Veenstra, and Antonius H. N. Cillessen. 2012. "Testing Three Explanations of the Emergence of Weapon Carrying in Peer Context: The Roles of Aggression, Victimization, and the Social Network." *Journal of Adolescent Health* 50:371–76.
- Doreian, Patrick. 1980. "Linear Models with Spatially Distributed Data: Spatial Disturbances or Spatial Effects?" *Sociological Methods and Research* 9:29–60.
- Doreian, Patrick. 1981. "Estimating Linear Models with Spatially Distributed Data." Pp. 359–88 in *Sociological Methodology*, vol. 12, edited by S. Leinhardt. San Francisco, CA: Jossey-Bass Publishers.
- Doreian, Patrick, and Frans Stokman. 1997. *Evolution of Social Networks*. London: Routledge.
- Dow, Malcolm M., Michael L. Burton, and Douglas R. White. 1982. "Network Autocorrelation: A Simulation Study of a Foundational Problem in Regression and Survey Research." *Social Networks* 4:169–200.
- Driver, Charles C., Johan H. L. Oud, and Manuel C. Voelkle. 2017. "Continuous Time Structural Equation Modeling with R Package ctsem." *Journal of Statistical Software* 77:1–35.
- Elhorst, J. Paul. 2001. "Dynamic Models in Space and Time." *Geographical Analysis* 33:119–40.
- Felmlee, Diane H., and David F. Greenberg. 1999. "A Dynamic Systems Model of Dyadic Interaction." *Journal of Mathematical Sociology* 43:497–523.
- Flache, Andreas, Michael Mäs, Thomas Feliciani, Edmund Chattoe-Brown, Guillaume Deffuant, Sylvie Huet, and Jan Lorenz. 2017. "Models of Social Influence: Towards the Next Frontiers." *Journal of Artificial Societies and Social Simulation* 20:2.
- Flashman, Jeniffer. 2012. "Academic Achievement and Its Impact on Friend Dynamics." *Sociology of Education* 85:61–80.
- Fouque, Jean-Pierre, George Papanicolaou, and K. Ronnie Sinclair. 2000. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge, England: Cambridge University Press.
- French, John R. P. 1956. "A Formal Theory of Social Power." *Psychological Review* 63: 181–94.
- Friedkin, Noah. E. 1998. *A Structural Theory of Social Influence*. Cambridge, England: Cambridge University Press.

- Gesell, Sabina B., Eric Tesdahl, and Eileen Ruchman. 2012. "The Distribution of Physical Activity in an After-school Friendship Network." *Pediatrics* 129:1064–71.
- Greenan, Charlotte C. 2015. "Diffusion of Innovations in Dynamic Networks." *Journal of the Royal Statistical Society, Series A* 178:147–66.
- Hamerle, Alfred, Hermann Singer, and Willi Nagl. 1993. "Identification and Estimation of Continuous Time Dynamic Systems with Exogenous Variables Using Panel Data." *Econometric Theory* 9:283–95.
- Haye, Kayla de la, Garry Robins, Philip Mohr, and Carlene Wilson. 2011. "Homophily and Contagion as Explanations for Weight Similarities among Adolescent Friends." *Journal of Adolescent Health* 49:421–27.
- Hays, Jude C., Aya Kachi, and Robert J. Franzese Jr. 2010. "A Spatial Model Incorporating Dynamic, Endogenous Network Interdependence: A Political Science Application." *Statistical Methodology* 7:406–28.
- Holland, Paul W., and Samuel Leinhardt. 1977. "A Dynamic Model for Social Networks." *Journal of Mathematical Sociology* 5:5–20.
- Huisman, Mark, and Christian Steglich. 2008. "Treatment of Nonresponse in Longitudinal Network Studies." *Social Networks* 30:297–308.
- Iacus, Stefano M. 2008. *Simulation and Inference for Stochastic Differential Equations. With R Examples*. New York: Springer.
- Knecht, Andrea, William J. Burk, Jeroen Weesie, and Christian E. G. Steglich. 2011. "Friendship and Alcohol Use in Early Adolescence: A Multilevel Social Network Approach." *Journal of Research on Adolescence* 21:475–87.
- Koskinen, Johan H., and Tom A. B. Snijders. 2007. "Bayesian Inference for Dynamic Social Network Data." *Journal of Statistical Planning and Inference* 137:3930–38.
- Krivitsky, Pavel N., and Mark S. Handcock. 2014. "A Separable Model for Dynamic Networks." *Journal of the Royal Statistical Society, Series B* 76:29–46.
- Leenders, Roger T. A. J. 1997. "Longitudinal Behavior of Network Structures and Actor Attributes: Modeling Interdependence of Contagion and Selection." Pp. 165–84 in *Evolution of Social Networks*, edited by P. Doreian and F. N. Stokman. Amsterdam, Netherlands: Routledge.
- Leenders, Roger T. A. J. 2002. "Modeling Social Influence through Network Autocorrelation: Constructing the Weight Matrix." *Social Networks* 24:21–47.
- Lomi, Alessandro, Tom A. B. Snijders, Christian E. G. Steglich, and Vanina J. Torlò. 2011. "Why Are Some More Peer Than Others? Evidence from a Longitudinal Study of Social Networks and Individual Academic Performance." *Social Science Research* 40:1506–20.
- Lospinoso, Josh A. 2012. "Statistical Methods for Social Network Dynamics." PhD dissertation, University of Oxford.
- Manski, Charles F. 1993. "Identification of Endogenous Social Effects: The Reflection Problem." *The Review of Economic Studies* 60:531–42.
- McFadden, Daniel. 1974. "Conditional Logit Analysis of Qualitative Choice Behavior." Pp. 105–42 in *Frontiers in Economics*, edited by P. Zarembka. New York: Academic Press.
- Mikosch, Thomas. 1998. *Elementary Stochastic Calculus with Finance in View*. Singapore: World Scientific Publishing.

- Mouw, Ted. 2006. "Estimating the Causal Effect of Social Capital: A Review of Recent Research." *Annual Review of Sociology* 32:79–102.
- Nicholson, Jody S., Pascal Deboeck, Jaelyn R. Farris, Steven M. Boker, and John G. Borkowski. 2011. "Maternal Depressive Symptomatology and Child Behavior: Transactional Relationship with Simultaneous Bidirectional Coupling." *Developmental Psychology* 47:1312–23.
- Nielsen, François, and Rachel A. Rosenfeld. 1981. "Substantive Interpretations of Differential Equation Models." *American Sociological Review* 46:159–74.
- Niezink, Nynke M. D., and Tom A. B. Snijders. 2017. "Co-evolution of Social Networks and Continuous Actor Attributes." *The Annals of Applied Statistics* 11: 1948–73.
- Norris, James R. 1997. *Markov Chains*. New York: Cambridge University Press.
- Øksendal, Bernt. 2000. *Stochastic Differential Equations. An Introduction with Applications*. New York: Springer Verlag.
- O'Malley, A. James. 2013. "The Analysis of Social Network Data: An Exciting Frontier for Statisticians." *Statistics in Medicine* 32:539–55.
- Oravecz, Zita, Francis Tuerlinckx, and Joachim Vandekerckhove. 2011. "A Hierarchical Latent Stochastic Differential Equation Model for Affective Dynamics." *Psychological Methods* 16:468–90.
- Ord, Keith. 1975. "Estimation Methods for Models of Spatial Interaction." *Journal of the American Statistical Association* 70:120–26.
- Oud, Johan H. L., and Marc J. M. H. Delsing. 2010. "Continuous Time Modeling of Panel Data by Means of SEM." Pp. 201–44 in *Longitudinal Research with Latent Variables*, edited by K. van Montfort, H. Oud, and A. Satorra. New York: Springer.
- Oud, Johan H. L., Henk Folmer, Roberto Patuelli, and Peter Nijkamp. 2012. "Continuous-Time Modeling with Spatial Dependence." *Geographical Analysis* 44: 29–46.
- Oud, Johan H. L., and Robert A. R. G. Jansen. 2000. "Continuous Time State Space Modeling of Panel Data by Means of SEM." *Psychometrika* 65:199–215.
- Pearson, Michael, and Lyann Mitchell. 2000. "Smoke Rings: Social Network Analysis of Friendship Groups, Smoking, and Drug-Taking." *Drugs: Education, Prevention and Policy* 7:21–37.
- Pearson, Michael, Christian E. G. Steglich, and Tom A. B. Snijders. 2006. "Homophily and Assimilation among Sport-Active Adolescent Substance Users." *Connections* 27: 51–67.
- Pearson, Michael, and Patrick West. 2003. "Drifting Smoke Rings: Social Network Analysis and Markov Processes in a Longitudinal Study of Friendship Groups and Risk-Taking." *Connections* 25:59–76.
- Reinecke, Jost, Peter Schmidt, and Stefan Weick. 2005. "Dynamic Modeling with Structural Equations and Stochastic Differential Equations: Applications with the German Socioeconomic Panel." *Quality and Quantity* 39:483–506.
- Ripley, Ruth M., Tom A. B. Snijders, Zsófia Boda, András Vörös, and Paulina Preciado. 2018. *Manual for RSiena (Version August 2018)*. Oxford: University of Oxford, Department of Statistics, Nuffield College; Groningen: University of Groningen, Department of Sociology.

- Robbins, Herbert, and Sutton Monro. 1951. "A Stochastic Approximation Method." *The Annals of Mathematical Statistics* 22:400–407.
- Robins, Garry, and Philippa Pattison. 2001. "Random Graph Models for Temporal Processes in Social Networks." *Journal of Mathematical Sociology* 25:5–41.
- Rosenberg, Morris. 1941. *Society and the Adolescent Self-Image*. Princeton, NJ: Princeton University Press.
- Rosenfeld, Rachel A., and François Nielsen. 1984. "Inequality and Careers: A Dynamic Model of Socioeconomic Achievement." *Sociological Methods and Research* 12: 279–321.
- Sacerdote, Bruce. 2014. "Experimental and Quasi-Experimental Analysis of Peer Effects: Two Steps Forward?" *Annual Review of Economics* 6:253–72.
- Schaefer, David R. 2016. "A Network Analysis of Factors Leading Adolescents to Befriend Substance-Using Peers." *Journal of Quantitative Criminology* 34:275–312.
- Schweinberger, Michael, and Tom A. B. Snijders. 2007. "Markov Models for Digraph Panel Data: Monte Carlo-Based Derivative Estimation." *Computational Statistics and Data Analysis* 51:4465–83.
- Shannon, Claude E. 1984. "A Mathematical Theory of Communication." *Bell System Technical Journal* 27:379–423, 623–56.
- Singer, Hermann. 1998. "Continuous Panel Models with Time Dependent Parameters." *Journal of Mathematical Sociology* 23:77–98.
- Singer, Hermann. 2012. "SEM Modeling with Singular Moment Matrices. Part 2: ML-Estimation of Sampled Stochastic Differential Equations." *Journal of Mathematical Sociology* 36:22–43.
- Snijders, Tom A. B. 2001. "The Statistical Evaluation of Social Network Dynamics." Pp. 361–95 in *Sociological Methodology*, vol. 31, edited by M. E. Sobel and M. P. Becker. Boston, MA: Blackwell Publishers.
- Snijders, Tom A. B. 2004. "Explained Variation in Dynamic Network Models." *Mathématiques, Informatique et Sciences Humaines* 168:31–41.
- Snijders, Tom A. B. 2005. "Models for Longitudinal Network Data." Pp. 215–47 in *Models and Methods in Social Network Analysis*, edited by P. J. Carrington, J. Scott, and S. S. Wasserman. New York: Cambridge University Press.
- Snijders, Tom A. B. 2017a. "Siena Algorithms." Technical report, University of Oxford and University of Groningen. https://www.stats.ox.ac.uk/~snijders/siena/Siena_algorithms.pdf.
- Snijders, Tom A. B. 2017b. "Stochastic Actor-Oriented Models for Network Dynamics." *Annual Review of Statistics and Its Application* 4:343–63.
- Snijders, Tom A. B., Johan Koskinen, and Michael Schweinberger. 2010. "Maximum Likelihood Estimation for Social Network Dynamics." *The Annals of Applied Statistics* 4:567–88.
- Snijders, Tom A. B., and Christian E. G. Steglich. 2015. "Representing Micro-Macro Linkages by Actor-Based Dynamic Network Models." *Sociological Methods and Research* 42:222–71.
- Snijders, Tom A. B., Christian E. G. Steglich, and Michael Schweinberger. 2007. "Modeling the Co-evolution of Networks and Behavior." Pp. 41–71 in *Longitudinal Models in the Behavioral and Related Sciences*, edited by K. van Montfort, J. Oud, and A. Satorra. New York: Cambridge University Press.

- Sørensen, Aage B. 1978. "Mathematical Models in Sociology." *Annual Review of Sociology* 4:345–71.
- Sørensen, Aage B. 1996. "Educational Opportunities and School Effects." Pp. 207–25 in *James S. Coleman*, edited by J. Clark. London: Falmer.
- Steele, J. M. 2001. *Stochastic Calculus and Financial Applications*. New York: Springer-Verlag.
- Steele, Joel S., Emilio Ferrer, and John R. Nesselroade. 2014. "An Idiographic Approach to Estimating Models of Dyadic Interactions with Differential Equations." *Psychometrika* 97:675–700.
- Steglich, Christian E. G., Tom A. B. Snijders, and Michael Pearson. 2010. "Dynamic Networks and Behavior: Separating Selection from Influence." Pp. 329–92 in *Sociological Methodology*, vol. 40, edited by T. F. Liao. Hoboken, NJ: Wiley-Blackwell.
- Steglich, Christian E. G., Tom A. B. Snijders, and Patrick West. 2006. "Applying SIENA: An Illustrative Analysis of the Coevolution of Adolescents' Friendship Networks, Taste in Music, and Alcohol Consumption." *Methodology* 2:48–56.
- Van Duijn, Marijtje A. J., Tom A. B. Snijders, and Bonne J. H. Zijlstra. 2004. "p2: A Random Effects Model with Covariates for Directed Graphs." *Statistica Neerlandica* 58:234–54.
- Voelkle, Manuel C., Johan H. L. Oud, Eldad Davidov, and Peter Schmidt. 2012. "An SEM Approach to Continuous Time Modeling of Panel Data: Relating Authoritarianism and Anomia." *Psychological Methods* 17:176–92.
- Wasserman, Stanley S. 1977. "Stochastic Models for Directed Graphs." PhD dissertation, Harvard University.
- Zhu, Xuening, Rui Pan, Guodong Li, Yewen Liu, and Hansheng Wang. 2017. "A Family of Association Coefficients for Metric Scales." *The Annals of Statistics* 45: 1096–123.

Author Biographies

Nynke M. D. Niezink is an assistant professor in the Department of Statistics and Data Science at Carnegie Mellon University. She received her PhD in the Department of Sociology at the University of Groningen. Her research focuses on social network methodology.

Tom A. B. Snijders is a professor of statistics and methodology in the social sciences in the Department of Sociology at the University of Groningen and an emeritus fellow of Nuffield College, University of Oxford. His research concentrates on multilevel analysis and social network analysis.

Marijtje A. J. van Duijn is an associate professor in the Department of Sociology at the University of Groningen. Her research interests include social network methodology and applied statistics.